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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1377

THE THEORIES OF TURBULENCE

By L. Agostini and J. Bass

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PREFACE

The theory of turbulence has made so much progress during these last years that it is of interest to state exactly the obtained results, the hypotheses on which these results are based, and the directions in which new research is being conducted.

Messrs. Bass and Agostini have undertaken this work and have summarized our actual knowledge of turbulence in a series of conferences which took place at the Sorbonne, within the Paris Institute of Mechanics. The reader will find in the following pages the text of these conferences, perfected and revised by Mr. Bass.

In view of the magnitude of the subject and its simultaneously physical and theoretical aspects it had seemed advisable to entrust this work to a team formed by a mathematician and a physicist. Mr. Bass had taken the responsibility for the theoretical part, Mr. Agostini for the physical part.

Initially, the report was intended to contain three theoretical and two physical chapters followed by a chapter on the technique of the measurements and on the appratus used: anemometers and statistical-measurement apparatus.

The unexpected death of Mr. Agostini in August 1949 unfortunately made modifications of the original project necessary. This premature death deprived us of a highly valuable physicist, at the peak of intellectual maturity, whose current work on these problems showed particularly remarkable promise.

Mr. Agostini had only just begun drawing up the two last chapters; Mr. Bass had to take up the editorial work and to complete it according to Mr. Agostini's notes. The chapter on the technique of the measurements has been omitted and will form the object of a later publication.

In order to make up for the gap in the experimental part, the text was supplemented by some curves furnished by Mr. Favre which will allow utilization for numerical calculations and will enable the reader to judge the agreement between theory and tests.

A. Fortier,
Professor at the Sorbonne

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THE THEORIES OF TURBULENCE*

By L. Agostini and J. Bass

INTRODUCTION

The theory of turbulence reached its full growth at the end of the 19th century as a result of the work by Boussinesq (1877) and Reynolds (1893). It then underwent a long period of stagnation which ended under the impulse given to it by the development of wind tunnels caused by the needs of aviation. Numerous researchers, mathematicians, aerodynamicists, and meteorologists attempted to put Reynolds' elementary statistical theory in a more precise form, to define the fundamental quantities, to set up the equations which connect them, and to explain the peculiarities of turbulent flows. This second period of the science of turbulence ended before the war and had its apotheosis at the 1938 Congress of Applied Mechanics.

During the war, some isolated scientists - von Weizsäcker and Heisenberg in Germany, Kolmogoroff in Russia, Onsager in the U.S.A. - started a program of research which forms the third period. By a system of assumptions which make it possible to approach the structure of turbulence in well-defined limiting conditions quantitatively, they obtained a certain number of laws on the correlations and the spectrum. These results, once they became known, caused a spate of new researches, the most outstanding of which are those by the team Batchelor-Townsend at Cambridge.

The analysis of these works became the subject of a series of lectures at the Sorbonne in February-March 1949, which subsequently were edited and completed. The mathematical theory of turbulence had already been published in 1946 (ref. 3) but practically ignored all publications later than 1940. Since the late reports have improved the mathematical language of turbulence, it was deemed advisable to start with a detailed account of the mathematical methods applicable to turbulence, inspired at first by the work of the French school, above all for the basic principles, then the work of foreigners, above all for the theory of the spectrum.

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*"Les Théories de la Turbulence." Publications Scientifiques et Techniques du Ministère de L'Air, No. 237, 1950.

The first chapter deals with the precise and elementary definition of turbulence (sections 1 and 2) and describes the tools of mathematical statistics on which the ultimate developments are based. Starting from paragraph 3, chapter I, the reader should be familiar with the definitions of the calculus of probabilities, the theorem of total probabilities, and the theorem of compound probabilities. This chapter is entirely theoretical, and its aim is to review the methods suggested by the theory of random functions and which seem likely to be applied to turbulence. Only the use of Navier's equations has, so far, produced positive results, and chapters III, IV, and V are largely devoted to it. However, it should be pointed out that there are theories less familiar to hydrodynamicists which have been proved in other branches of physics (kinetic theory of gases, quantum mechanics). The purely random method is described in paragraph 5, and its adaptation to molecular systems (according to Born and Green) in paragraph 7. The statistical method has the advantage of furnishing a remarkable demonstration of the general equations of hydrodynamics (paragraph 6) and of providing an exact classification of the statistical parameters of turbulence (paragraph 4), which is interesting to keep in mind when studying the foreign reports, too exclusively devoted to spatial correlations. In any case, the reader who wants to read chapters IV and V can pass up most of chapter I, except perhaps paragraphs 1 and 2, without major trouble.

Chapter II deals with the kinematics of statistical mediums and, particularly, isotropic mediums. It seemed practical to include at the same time the velocity correlations, the theory of which has been given in almost final form by Kármán, at the end of the second period, and of the spectrum, the theory of which, due to Taylor's initiative, has only been achieved very recently. Only paragraphs 10 and 11 refer to statistical functions, and their detailed knowledge is not indispensable for reading the rest of the chapter. The results and the formulas of chapter II are constantly applied in the subsequent chapters, but it is not necessary to know the proofs which are, in most cases, a simple matter of calculation. The most important of these formulas are, moreover, compiled in a special section following chapter V.

Chapter III is a mathematical study of the application of Navier's equations to turbulent motion. Paragraphs 17, 18, 19, and 22 are fundamental. Their main purpose is to recall Kármán's results of 1938 with some improvements and some supplements of more recent date. The paragraphs 20 and 21 review a certain number of physically reasonable solutions of the Kármán-Howarth fundamental equation. Some of these assume particular importance in chapter V but, first, it seems advisable to give an impartial view of the whole and to proceed progressively into the domain of the concrete. The paragraphs 23 and 24 deal with an equation by Heisenberg which involves time correlations and from which probably not all possible results have been extracted. It is not indispensable to have knowledge of this in order to continue. Paragraph 25 contains a

mathematical account of Heisenberg's numerical theory of the spectrum, which is taken up again in chapter V from a more physical point of view and whose examination, contrary to paragraphs 23 and 24, proved useful before attacking chapter V.

Chapters IV and V deal with new physical theories involving similarity hypotheses and producing numerical laws. Chapter IV reviews the works of Kolmogoroff and Weizsäcker, chapter V those of Heisenberg, Batchelor, and Townsend on the decay of turbulence created by grids.

Finally, in an appendix, the theoretical discussions of chapters III, IV, and V are illustrated by some correlation curves and spectrum curves measured directly in the wind tunnel by A. Favre, in the laboratory of the mechanics of the atmosphere at Marseille, or derived from experimental curves by elementary transformations.

An exhaustive study of modern theories of turbulence calls for some knowledge of the calculation of tensors, probabilities, and statistical analysis, besides the classical conceptions of differential and integral calculus.

As regards the tensors, knowledge of the definitions and fundamental operations with rectangular cartesian coordinates is sufficient. There are a number of articles on this subject, but they generally lean toward the tensor analysis with curvilinear coordinates for which there is no need. (It should be noted that the tensor analysis plays, in contrast, an important part in the theory of the boundary layer around an airfoil.) Incidentally, there is available a little book recently published, by Lichnérowicz, entitled: Elements of tensor calculus (collection Armand Colin). On mathematical statistics, the book by Darmois, published by Doin (1928), can be consulted. For the elementary theory of random functions, consult the first part of Bass' report (ref. 3) and the appendix to d'Angot's "complements of mathematics" (editions of the Revue d'Optique), edited by Blanc-Lapierre. More detailed information on statistical functions can be found in Levy's book: "Stochastic Processes and Brownian Motion" (1948).

The present report deals only with general theories which are valid whatever the physical or geometric causes of turbulence may be, as is shown in chapter IV. Only in chapter V the assumptions are limited and the study involves a problem of decay of turbulence that is compatible with the turbulence in wind tunnels. These theories are probably applicable to mediums of extremely diverse scales, from the microturbulence to the terrestrial atmosphere (Dedebant and Wehrlé), to stellar atmospheres and to interstellar matter (Weizsäcker). However, among the hypotheses there is always that of the incompressibility, which precludes the application to sonic or supersonic flows. This is not an indispensible hypothesis, but it simplifies the calculations considerably, and the

consequences are easy to check. So the incompressible turbulent motions are practically the only ones studied up to now.

Among the other hypotheses worthy of discussion, those referring to processes of energy transport (paragraph 24 and chapter V) assume that all turbulent energy comes from the motion of the whole, dissociated and broken up by the obstacles of periodic structure. But nature furnishes examples of different turbulences such as the so-called thermal turbulence, for example, where the source of energy is not the fan of a wind tunnel but the solar radiation suitably transformed into kinetic energy. Kolmogoroff's theory of local isotropy applies probably to thermal turbulence, but the forms of energy transport in the spectrum must be different from those encountered in wind tunnels. The same applies to the astronomic turbulent mediums alluded to previously. So the foregoing remarks limit the scope of the recent theories, in a certain measure.

Omitted entirely was the problem of turbulent boundary layer which, experimentally, depends on the same technique, but has been approached by different mathematical methods. Furthermore, it is a complicated problem where the turbulence is neither homogeneous nor isotropic.

In what measure are the results, suitably demarcated, defined? The statistical character of the turbulent velocity seems a clear and well-established notion and consequently the mechanics of turbulence will be a statistical mechanics. As far as the kinematics (chapter II) are concerned, we are therefore on solid ground. The dynamics of turbulence (chapter III) itself is likewise well established, by means of the hypothesis of the validity of the Navier equations. This hypothesis, generally adopted because it is convenient and, one might say, necessary, has however at times raised considerable doubts. However, while experimental verifications do not contradict it, there is yet no occasion to reject it.

But, what should be expected from experimental verifications? First of all, it is found that the theories are still rather imperfect. In fact, the theories are usually limited, acceptable in the limiting conditions which are difficult to attain actually (very high Reynolds numbers, for example). No rigorous verification should be expected since the true conditions are too far removed from the theoretical conditions.

On the other hand, the accuracy of measurement is low. The original reason for it lies in the very nature of the turbulent phenomenon, and its irregular and badly defined character. Furthermore, the anemometers are coarse instruments, their operation not sufficiently known in the presence of turbulence, and their interference with the fluid sometimes a little mysterious. All this helps to lower the experimental precision. To measure the "length of dissipation," for example, two ways are open: one is to measure the correlations of the velocity at two infinitely close

points which has no practical sense since two anemometers cannot be brought together indefinitely; the other is to use one anemometer for measuring the mean square value of the derivative of velocity. But the operation of taking a derivative of a function as complicated as the velocity is by itself inaccurate and introduces serious scattering of the points of the derivative curve. So in both cases, the accuracy of the results is extremely limited.

In consequence, the experimental verifications can be applied only to the orders of magnitude. To illustrate: to check whether a parameter, according to theory, is constant, one constructs the representative curve of the parameter and, if this curve has a sufficiently extended maximum, one estimates that the experiment closely confirms the theory, conceding that, as the experimental conditions more nearly approach those stipulated by theory, the maximum flattens out more and more, and one does not appear too severe in the examination of the scatter of the test points.

On this assumption, the verifications of the experimental laws of turbulence, due in particular to Townsend, are encouraging. Therefore, it is well to retain the hypotheses of chapters III, IV, and V, although some of them obviously have their limitations, and we ought not hope for results greater than actually can be given. Take an example drawn from the theory of the spectrum, for instance. At the beginning of decay of turbulence in a tunnel and at high Reynolds numbers, a certain spectral function $F(k)$ is of the form Ck^4 for the small values of wave number k. It then passes through a maximum, then becomes proportional to $k^{-5/3}$, and finally approaches zero as $k \longrightarrow \infty$, maybe as k^{-7}. But the regions of the axis of k in which the fragments of the laws to be enumerated remain acceptable are badly defined and connected by zones of which the structure is not known. So the future task of the theorists will be to combine these partial results into a single acceptable law at least to the extent that it does not become fundamentally incompatible with the nature of turbulence, since $k = 0$ up to $k = \infty$. Only then will there be a true theory of turbulence.

CHAPTER I

GENERAL ASPECTS OF TURBULENCE

THE STATISTICAL METHOD

1. Definition of turbulence:

Our purpose is to discuss the operation which consists of measuring, at a given point, the velocity of a flow which, to avoid every difficulty, is assumed to be steady and uniform at the usual macroscopic scale. The measuring instrument is an "anemometer" with approximately determined dimensions and time constant, and of sufficiently ideal nature so as not to disturb the flow by its presence. If this anemometer is large enough, it measures the "velocity of the main flow" of the fluid. If it is very small, it can be imagined that it operates in discontinuous manner, never undergoing the influence of more than one molecule at a time. An anemometer sensitive to the individual action of molecules is, of course, unattainable, but the idea of such an instrument is convenient for representing the extreme limit of fineness of kinematic measurements in a fluid. Between these two extremes, the indications of the anemometer depend upon the structure of the fluid. It may happen, by exception, that, when its dimensions are progressively reduced, the velocity which it indicates remains unchanged up to the moment where the individual influence of the molecules starts to make itself felt and where the indications lose all statistical significance. The flow is then said to be laminar.

But, in general, the matters are otherwise. We start with a first anemometer which, through its dimensions, fixes a certain scale of measurement. This anemometer measures the mean speed of the molecules in a certain volume V. Then it is replaced successively by smaller anemometers in such a way that volume V decreases progressively. It happens that, from a certain value V_1 of V, the numerical indication supplied by the anemometer changes. If V is decreased continuously, the new indication remains stable up to a certain value V_2, then changes again and so on. The intervals (V_1, V_2), (V_2, V_3), . . . characterize the various scales of turbulence, and the motion of the fluid is said to be turbulent. More exactly, they are the conditions necessary for a fluid to be turbulent, and which must be defined and perfected to make them sufficient and practical.

The last value V_n of the series $V_1, V_2, \ldots$ is that from which onward, the notion of average loses its significance, the number of molecules contained in the volume V_n not being large enough any longer for statistical purposes. The series $V_1, V_2, \ldots$ can be discrete or

continuous. If it is discrete, it still does not imply that the critical values V_1, V_2, . . . are mathematically defined. In the vicinity of V_1, a rapid variation of the anemometer indications occurs, which subsequently become stabilized in the region (V_1, V_2), and so forth. If this stabilization is not very clear, the turbulent scales are said to succeed one another in continuous fashion.

The previous discussion ends with the notion of the turbulent fluid. But the definition of the laminar motion given above is a little too restrictive and the distinction between laminar and turbulent still not precise enough, as proved by the following example:

Consider the motion of air produced by stationary waves in a sound tube. The motion of the whole reduces, at rest, to large scale. But, at each point there exists a speed other than zero, a periodic time function which, at a given instant, varies periodically from one point to another. This motion, lying between the system at rest and the molecular agitation, has not a turbulent character.

Turbulence, as shown, implies first the notion of scale. But it should be added that, at a given scale, each component of the velocity at a point is a function of time presenting a character of periodicity without fundamental period. This is not a periodic function but a sum of harmonics, the frequencies of which are not multiples of an identical fundamental frequency. This irregularity of the turbulent agitation is essential and distinguishes it from sound agitation, or preturbulent vortical motions, like the cellular vortices of Bénard. The mathematical symbol for the turbulent velocity is not the ordinary Fourier series, but Fourier's integral. It will be discussed later.

This concept of irregular agitation at a point as function of the time is not itself sufficient. It makes it possible to differentiate the turbulent agitation from the periodic sound agitation (musical sound), but not from the noise, which is an agitation without definite period. What distinguishes the noise from turbulence is the fact that it is propagated by waves that exist on surfaces of equal phase, and consequently have a regular spatial distribution, notwithstanding the irregularity in the time of the local velocity.[1]

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[1]A descriptive and purely kinematic distinction is involved here. Its cause (compressibility) is not discussed.

The difference between turbulence and sound agitation should become plain from the following example: In a turbulent wind tunnel, we select at a point the longitudinal component of the velocity with a hot wire and send the electric current of the hot wire to a loud speaker on the outside of the tunnel. The atmosphere becomes the source of an irregular agitation, which propagates by waves and is not turbulence, although, at a point, the internal turbulent motion and the external sound motion have some important kinematic elements in common.

For a given scale, each component of the turbulent velocity is an irregularly periodic function of both space and time.

It seems that turbulence is well defined by these kinematic conditions and hence is distinguished from all other more organized fluid motions.

2. Average values - statistics:

Figure 1 represents the record of a component u of the turbulent velocity as function of the time. (Record of the velocity of turbulent agitation in a 20-cm by 30-cm tunnel: airspeed, 20 m/sec; intensity of turbulence, $5\text{-}10^{-3}$; time of recording is 0.03 second.)

The most natural method of measuring the mean velocity on the graph consists in forming the ordinary integral

$$u_m = \frac{1}{T}\int_0^T u(t)\,dt \tag{2-1}$$

extended over the total duration T of recording. This method is, in general, not very satisfactory because the operation lacks precision when the curve $u(t)$ is complicated.

A more precise method consists in dividing the graph by parallels to the axis of t, suitably close together in the ordinates u_1, u_2, u_3, . . ., in measuring the number n_i of points where the line of the ordinate $\frac{1}{2}(u_i + u_{i+1})$ meets the curve $u(t)$ and then in computing the quantity

$$\bar{u} = \sum_i u_i \frac{n_i}{n} \tag{2-2}$$

$n = \sum_i n_i$ is the total number of points met by all parallels.

For this calculation, a profitable first stage consists in first constructing the graph giving the corresponding statistical frequency $f_i = \frac{n_i}{n}$ for each velocity u_i. Crossing the limit obviously makes it possible to plot a curve of frequency $f = f(u)$ (fig. 2) such that the proportion of the values of the velocity comprised between u and $u + du$

is equal to $f(u)du$, the integral $\int_0^\infty f(u)du$, which replaces $\sum \frac{n_i}{n}$, being equal to unity. The final expression of $\bar{u}$ is then

$$\bar{u} = \int_{-\infty}^{\infty} uf(u)du \tag{2-3}$$

It replaces (2-2) and should be compared with (2-1).

The practical operations enabling the replacement of u_m by $\bar{u}$ correspond to well-known mathematical operations. The mode of computing $\bar{u}$ is that of a Lebesgue integral, and u_m is an integral of the classical type of Riemann. If u_m is computable, both methods yield the same result. But it may happen that Riemann's integral does not exist because the function $u(t)$ is too complicated mathematically. However, in general, Lebesgue's integral exists (if $u(t)$ is measurable, and naturally bounded). This mathematical case corresponds to the practical case where the curve $u(t)$ is too complicated for an accurate continuation of the integration. The function $f(u)$, continuous and differentiable in the current cases, is the medium which, determined once for all, replaces the calculation of Lebesgue's integral by that of an ordinary integral by means of the plotting of curve $f(u)$.

From the function $f(u)$, other averages can be computed. For example, the amount of differences of the speed with respect to its mean value can be figured by computing the mean value of $(u - \bar{u})^2$. Rather than defining this average by Riemann's integral

$$\frac{1}{T} \int_0^T \left[u(t) - u_m\right]^2 dt \tag{2-4}$$

it is simpler and more precise to use the formula

$$\overline{(u - \bar{u})^2} = \int_{-\infty}^{\infty} (u - \bar{u})^2 f(u)du \tag{2-5}$$

which, once the curve $f(u)$ is plotted, calls only for operations of a simple character.

Thus, it is seen how much the construction of the curve $f(u)$ simplifies the numerical calculations of turbulence. Various experimental techniques make for direct attainment of this curve without passing through the numerical analysis of a velocity record. For measuring the averages such as $\overline{(u - \bar{u})^2}$, it is often advisable also to use specialized equipment without first plotting the curve $f(u)$.

3. Random variables and the laws of probability:

The theoretical significance of the function $f(u)$ is analyzed. It groups the statistical data contained in the initial curve $u(t)$, with this exception that the chronological order in which the velocities actually follow one another does no longer appear. This limitation is quite natural, though, and it will be seen later that this order reappears, in a certain measure, by the introduction of space and time correlations.

Obviously, only statistical data can supply stable information on turbulence. When the same record of the velocity is begun again several times while taking every reasonable precaution so that the conditions are identical, it obviously results in curves $u(t)$ which absolutely are not superposable. The function $u(t)$ has not, therefore, the character of permanence that is suitable for representing the laws of a physical phenomenon. But this character is relevant to the function $f(u)$ or to averages such as $\bar{u}$, $\overline{(u - \bar{u})^2}$ whose values are characteristic numbers of the investigated flow, and which are derived by simple mathematical operations from the function $f(u)$. Hence, we direct our attention to this function which can be regarded as representing the first law of turbulence.

To say that the velocity is characterized by a curve of statistical frequencies is to say, by comparing the frequencies with probabilities and $f(u)$ with a density of probability, that this velocity is a statistical quantity; $f(u)du$ is the probability that the chosen component of the velocity is contained between u and $u + du$.

A priori, such a law of probability could be dependent on the time. That would correspond to a turbulent flow for which the laws varied with respect to time. There is no contradiction to the initial assumptions of permanence here. It is a question of scale. In order for the experimental operation by which $f(u)$ is defined to have any meaning, it is necessary that two conditions be realized simultaneously:

(1) The number of oscillations in the time interval T involved must be great enough to furnish satisfactory statistics.

(2) The laws of turbulence in this time interval T must be practically permanent.

If it is not so, it might be difficult to reconcile the statistical theory with the experiment. Fortunately, those "ergodic" conditions are practically always realized in the usual cases and are therefore taken for granted in the following:

The velocity has three components u_1, u_2, and u_3 which are treated as three random variables, components of a random vector.

It should be noted here that the velocity is perhaps not sufficient for characterizing the turbulence. It might appear useful to introduce other quantities, such as the pressure, which should be treated as a statistical quantity. But, owing to the equations of motion, this then will be a function defined by the velocity and its derivatives. For the present, it is assumed that the turbulent motion is sufficiently well defined by its velocity so that the problem narrows down to the laws of probability applied to the velocity.

The simplest of these laws, that which immediately generalizes the experimental function $f(u)$, is the law of probability of the system of three components u_1, u_2, u_3 of the velocity. This law may vary as function of the time t (problem of spontaneous decay of turbulence) and of the space (variation of turbulence in terms of the distance from the walls). It is therefore a function of t and the ordinates x_1, x_2, x_3 of the point of measurement. Its density is denoted by

$$f(u_1,\ u_2,\ u_3;\ x_1,\ x_2,\ x_3,\ t)$$

or, abbreviated, $f(u;\ x,\ t)$.

The quantity $f\ du_1\ du_2\ du_3$ or, abbreviated, $f\ du$, represents the probability that, at the point x_1, x_2, x_3 (or x) and at the instant t, the three velocity components are comprised between u_1 and $u_1 + du_1$, u_2 and $u_2 + du_2$, u_3 and $u_3 + du_3$.

But a single law of probability defined in terms of four parameters x_1, x_2, x_3, and t is not adequate for characterizing turbulence. It is necessary to introduce the more profound concept of random function and to consider the turbulent velocity as a random function of space and time. This is the random velocity field.

This point is now to be defined. An isolated random quantity is defined by its law of probability. But, to define a system of coexistant random quantities requires more than just their laws of individual probabilities. The stochastic dependencies or correlations between these

quantities must also be known. Taking a family of probability laws does not give the right to speak of the system of corresponding random quantities without completing the data by those of the correlations. Supposing that the family in question depends on a continuous parameter t. We know then an isolated random quantity $U(t)$ for each value of t. This immediately suggests grouping the $U(t)$ corresponding to the various values of t in a well-defined system of statistical quantities. This calls for the introduction of the correlations between the $U(t_1)$, $U(t_2)$, . . . corresponding to an arbitrary system $\underline{E}$ of the values t_1, t_2, . . . of the parameter. To proceed thus, means to define a random function. Naturally, this also holds for laws of probabilities dependent on several parameters. Thus, when a turbulent medium is represented as a velocity field, the system of velocities at each point and at each instant precisely constitutes a system of coexistant statistical quantities, of which the physical interactions characterizing the structure of turbulence have the correlations for mathematical description.

Among the systems $\underline{E}$, the simplest are the denumerable systems and even the finite systems and, among the latter, the simplest one which is not trivial is that of two elements, that is, of two points of space and time.

The concept of random function thus suggests the comparison of the velocity vectors at two different points of their field of definition, that is, for two different positions x and x' (x represents the point of the coordinates x_1, x_2, x_3) and for two instants t and t'. It concerns a statistical comparison which makes it possible to define the law of probability of the two systems of the velocity components at points x, x' and instants t and t'. This law has a very clear physical meaning and is easy to define experimentally or, at the least, to construct the surface of (statistical) frequencies corresponding to one velocity component at point x and a second component at point x', the measurements being spaced at a chosen time interval τ.

In practice, the question is frequently handled from a less general point of view. One is not concerned with the laws of probability themselves as such but only with their most simple moments, those of the second order which are associated, as will be shown, with certain "physical" aspects of turbulence and, possibly, with certain moments of the third order that play a part in modern theories. The moments of the second order constitute the correlation tensor of the law of probability of the velocity field at two points and at two instants. They are measured direct, without resorting to frequency curves or frequency surfaces or to velocity recordings. A detailed study follows later.

4. The concept of random point - velocity field - turbulent diffusion:

The density of probability $f(u; x, t)$ of the velocity field contains the velocity of the whole fluid, but not its density. With $\rho(x_1, x_2, x_3; t)$ or, abbreviated, $\rho(x, t)$ denoting the quotient of the density at point x by the fluid mass, ρ is a normalized function, as a density of probability, which means that

$$\int_V \rho \, dx = 1 \tag{4-1}$$

where dx represents the element of volume $dx_1 \, dx_2 \, dx_3$ and the integral is extended to the volume V occupied by the fluid.

In all modern studies on turbulence, the fluid is naturally assumed incompressible, so that ρ is a constant, equal to $1/V$ in the volume V, and zero at the outside. ρ could be simply replaced by a constant; but the more general conclusions to be arrived at ultimately are more complete if this simplification is not made. On the other hand, it is interesting to foresee, at a certain stage of the theory, the day when it will be possible to study the turbulent motions in conditions where the compressibility is no longer negligible. For these reasons, ρ is treated here as a function of x (and even of t, if necessary).

The product $R(x, u; t) = \rho f$ is now formed. It obviously is normalized with respect to the system of the six variables x, u

$$\int R \, dx \, du = 1 \tag{4-2}$$

R presents thus the characters of a density of probability with respect to these six variables. The quantities x_1, x_2, x_3 are regarded as the coordinates of a moving point, u_1, u_2, u_3 as the velocity components of this point. These are six random quantities of which the law of probability at instant t is known. Thus a random point can be associated with the turbulent fluid in correspondence with a given scale.

This point is now to be discussed as was the velocity field in the preceding paragraph.

The position and the velocity of this point are random functions of the time. The theory of random functions suggests the study of the law of probability of the system of positions and velocities of this point for an arbitrary combination of instants. The initial analysis was on

the law of probability at one instant. The generalization from one to two instants seems to us adequate for forming a physical theory of turbulence and, in particular, for considering the statistical organization in time of the velocity, and what may be called the "interactions between turbulent particles."

The values of the positions and of the velocity must therefore be associated to two instants t and t'. This association is a stochastic relationship defined by the density of probability of the following system of 12 statistical variables:

$$\left.\begin{array}{ll} x_1, x_2, x_3 & \text{position} \\ u_1, u_2, u_3 & \text{velocity} \end{array}\right\} \text{at instant } t$$

$$\left.\begin{array}{ll} x_1', x_2', x_3' & \text{position} \\ u_1', u_2', u_3' & \text{velocity} \end{array}\right\} \text{at instant } t'$$

or, abbreviated,

$$G(x, x', u, u'; t, t')$$

and it is assumed that it defines the turbulent motion. By what stages can it be measured?

According to the theorem of compound probabilities, $G\,dx\,dx'\,du\,du'$ is the product of two factors:

(1) The probability that at the instants t and t' the statistical image point of the fluid might have positions contained within the intervals $(x, x + dx)$ and $(x', x' + dx')$.

(2) The probability (conditional) that, these positions being fixed, the velocities are contained in the intervals $(u, u + du)$ and $(u', u' + du')$.

This last probability is designated by $H(u, u'; x, x', t, t')du\,du'$. According to the theorem of compound probabilities, the probability (1) is the product of the probability $\rho(x, t)dx$ that the random point is found, at instant t, in the interval $x, x + dx$, through the (conditional) probability that, the position at instant t being chosen, its position at instant t' is found in the interval $x', x' + dx'$.

This last probability is designated by $p(x'; x, t, t')dx'$. We can write

$$G(x, x', u, u', t, t') = H(u, u'; x, x', t, t')p(x'; x, t, t')\rho(x; t) \tag{4-3}$$

What is the significance of the three factors of which G is the product?

We already know ρ, which represents except for a numerical factor the density of the fluid.

The function H is the law of a random velocity field at two points and at two instants. It is natural to identify it with the law of the field already discussed at the end of paragraph 3. It is seen that it gives no complete picture of turbulence. It does not explain the function p.

The function p represents the turbulent diffusion at a chosen scale. It is the relative density at point x' and at instant t' of fluid elements which have passed neighboring point x at instant t. It should be pointed out that the diffused portion of the fluid is essentially compressible since the density $p(x'; x, t, t')$ decreases in proportion as the point x' is removed from the initial point x where it is maximum. The spread can be materialized and p can be measured by introducing with the necessary precautions at point x a dye that spreads in the fluid. But it is a rather ticklish matter to separate the effects of the various turbulent scales, especially of the molecular diffusion. It is accomplished by adapting the particles of the "dye" to the chosen scale. This way Kampé de Fériet rendered the turbulent diffusion of the tunnel flow visible in his experiments at the Institute of Fluid Mechanics, at Lille, by injecting soap bubbles at a point in place of dyed particles. These soap bubbles, because of their size, were sensitive to the turbulent fluctuations and were used successfully for measuring a density of turbulent diffusion.

5. Equations of development of the laws of probability:

The analysis of the turbulent velocity field made it possible to represent a turbulent fluid by two random functions $X(t)$ and $U(t)$ playing the part of the position and of the velocity for a random material point. It was shown how this concept of random point gives a very complete picture of the turbulence. But this picture is qualitative and must be made more quantitative.

At the beginning, no distinction is made between the velocity vector $X(t)$ and the position vector $U(t)$. The six components of these two vectors are considered as those of a unique vector $\underline{X}(t)$ in a six-dimensional space, by putting $u_1 = x_4$, $u_2 = x_5$, $u_3 = x_6$, and designating the density of probability of this vector by $R(x; t)$, this notation being the abbreviation of $R(x_1, x_2, x_3, x_4, x_5, x_6; t)$, that is, of $R(x_1, x_2, x_3, u_1, u_2, u_3; t)$.

We already had applied (4-3), the theorem of compound probabilities, to the law of probability G of the position and the velocity at two instants t, t':

$$G(x, x', u, u'; t, t') = \left[\rho(x; t)p(x'; x, t, t')\right] H(u, u'; x, x', t, t')$$

The question involved essentially the separation of the velocity, which figures in the factor H, from the position. But there is another way of applying this theorem. It consists in separating the two instants t and t' by writing

$$G(x, x', u, u'; t, t') = R(x', u'; t')K(x, u; x', u', t, t') \tag{5-1}$$

hence it results, according to the theorm of the total probabilities, that

$$R(x, u; t) = \int R(x', u'; t')K(x, u; x', u', t, t')dx'\,du' \tag{5-2}$$

K is a conditional density of probability, that of the "probability of passage" of the state of the random point at instant t' to its state at instant $t > t'$.

Abbreviated, we get

$$R(x; t) = \int R(x'; t')K(x; x', t, t')dx' \tag{5-3}$$

To exploit this equation, recourse is had to a method patterned after the concept of J. Moyal (ref. 33). This idea was to consider (5-3) as a linear integral transformation for passing from the density of probability R at instant t' to the same density at instant t. The density

of probability of the passage K is the kernel of the transformation. With $K_{tt'}$ indicating the linear operation which has as kernel the function $K(x; x', t, t')$, one may write symbolically:

$$R(x; t) = K_{tt'} R(x'; t') \tag{5-4}$$

This transformation has special properties rather difficult to define. Suffice it to state that it reduces to the identical transformation for $t = t'$. It is assumed that it has an inverse $K_{tt'}^{-1} = K_{t't}$ without, however, prejudicing the relations between $K_{tt'}$ and $K_{t't}$.

The infinitesimal transformation applied to $K_{tt'}$ is now examined. To this end, R is assumed differentiable with respect to t and $\partial R/\partial t$ calculated

$$\frac{\partial R(x, t)}{\partial t} = \lim_{\tau \to 0} \frac{R(x; t + \tau) - R(x; t)}{\tau}$$

$$= \lim_{\tau \to 0} \frac{K_{t+\tau t'} K_{t't} R(x', t')}{\tau}$$

$$= \lim_{\tau \to 0} \frac{K_{t+\tau t'} K_{t't} - 1}{\tau} R(x, t)$$

If, as assumed, the function R is differentiable with respect to t, the limit of the second member exists, and it is a function of x,t independent of t'. It can be computed by giving parameter t' (independent of variable τ) any value not exceeding t, such as $t' = t$, for example. Hence

$$\frac{\partial R}{\partial t} = \lim_{\tau \to 0} \frac{K_{t+\tau t} - 1}{\tau} R \tag{5-5}$$

The operator

$$L = \lim_{\tau \to 0} \frac{K_{t+\tau t} - 1}{\tau}$$

defines the infinitesimal transformation of K, and R confirms the fundamental functional equation

$$\frac{\partial R}{\partial t} = LR \tag{5-6}$$

which, theoretically, enables $R(x; t)$ to be computed at instant t when $R(x; t')$ is known at an initial instant t'.

The foregoing calculation cannot be explained in a simple manner from formula (5-3) because the limit of the function $K(x, x', t, t')$ does not exist when $t' \longrightarrow t$; this is a symbolical "function of Dirac" which expresses the identity $K_{tt} = 1$ in the functional formalism. It is preferable to pass, as Moyal did, from the densities of probability to characteristic functions. Moyal's calculation follows:

With the function K is associated the characteristic function of the increment $\underline{X}(t) - \underline{X}(t')$, the value of $\underline{X}(t')$ once fixed, or by definition

$$\Phi(\alpha; x', t, t') = \int e^{i\alpha(x-x')} K(x; x', t, t')dx \tag{5-7}$$

One assumes likewise:

$$\varphi(\alpha, x, t) = \int e^{i\alpha x} R(x; t)dx \tag{5-8}$$

the characteristic function of $\underline{X}(t)$.

According to the theorem of total probabilities, one has:

$$\varphi(\alpha, x, t) = \int e^{i\alpha x'} \Phi(\alpha; x', t, t')R(x'; t')dx' \tag{5-9}$$

This relation replaces (5-3).

The two members of (5-9) are differentiated with respect to t

$$\frac{\partial \varphi}{\partial t} = \lim_{\tau \to 0} \int e^{i\alpha x'} \frac{\Phi(\alpha; x', t + \tau, t') - \Phi(\alpha; x, t, t')}{\tau} R(x'; t')dx'$$

Whereas the conditions of differentiability under the $\int$ sign (permutation of signs lim and of $\int$) cannot be verified on (5-3), they can be here, in all current cases. Since the limit must not depend on t', let $t' = t$, so that

$$\frac{\partial \varphi}{\partial t} = \int e^{i\alpha x'} R(x', t) \lim_{\tau \to 0} \frac{\Phi(\alpha; x', t + \tau, t) - 1}{\tau} dx' \qquad (5\text{-}10)$$

because $\Phi(\alpha; x, t, t) = 1$, for $t' = t$.

The limit that figures under the $\int$ sign is a certain function $\theta(\alpha; x', t')$, whence follows the basic equation

$$\frac{\partial \varphi}{\partial t} = \int e^{i\alpha x'} R(x', t)\theta(\alpha; x', t)dx'$$

equivalent to (5-6). From that, two probability densities can be recovered by taking the Fourier transforms of the two members. Lastly, $\partial R/\partial t$ is expressed in form of an integral transformation of R, equivalent to the transformation L, which, after introducing a kernel function $L(x, x', t)$, gives

$$\frac{\partial R(x, t)}{\partial t} = \int R(x', t)L(x, x', t)dx' \qquad (5\text{-}11)$$

The second member of (5-11) is represented in integral form but, in many cases, it can be expressed in form of a differential operator of finite or infinite order.

Examples.- Supposing the probability of passage obeys the Laplace-Gauss law and, to avoid any confusion of the notations, t_0 now denotes the previous instant; the differences of $\underline{X}(t_0)$ and $\underline{X}(t)$ are denoted by $S_0 = S(t_0)$ and $S = S(t)$, the correlation coefficient between $\underline{X}(t_0)$ and $\underline{X}(t)$ by $r = r(t_0, t)$. It is known that the law of probability related to $\underline{X}(t)$, when the value x_0 of $\underline{X}(t_0)$ is given, has for density

$$K(x; x_0, t_0, t) = \frac{1}{\sqrt{2\pi}S\sqrt{1 - r^2}} e^{-\frac{1}{2(1-r^2)}\left[\frac{x}{s} - r\frac{x_0}{s_0}\right]^2} \qquad (5\text{-}12)$$

Using Moyal's method, it is shown that the probability density $R(x, t$ which assumes K as kernel of the probability of passage satisfies the equation of partial derivatives

$$\frac{\partial R}{\partial t} = - \left(r' + \frac{S'}{S} \right) \frac{\partial}{\partial x}(xR) - r'S \frac{\partial^2 R}{\partial x^2} \tag{5-13}$$

where r' represents the value of $\frac{\partial}{\partial t_1} r(t_1, t)$ for $t_1 = t$, and S' denotes the derivative $\frac{d}{dt} S(t)$.

The equation (5-13) is, moreover, demonstrated very simply by a direct method. By definition of K

$$R(x, t) = - \frac{1}{\sqrt{2\pi} S \sqrt{1 - r^2}} \int_{-\infty}^{\infty} R(x_0, t_0) e^{- \frac{1}{2(1-r^2)} \left[\frac{x}{s} - r \frac{x_0}{s_0} \right]^2} dx_0 \tag{5-14}$$

this equation being, in particular, satisfied when

$$R(x, t) = \frac{1}{S\sqrt{2\pi}} e^{- \frac{x^2}{2s^2}}$$

When the two members of (5-14) are differentiated with respect to t and x, equation (5-13) is verified. Reciprocally, the integral of (5-13) which is reduced to $R(x, t_0)$ for $t = t_0$, can be put in the form (5-14).

The construction of the random functions compatible with these laws of probability is an easy matter.

To illustrate:

Let $h(s)$ be a variable random function of the parameter s, obeying the reduced Laplace-Gauss law $\left(\overline{h} = 0, \overline{h^2} = 1 \right)$, and so that the increments

$h(s_1) - h(s_2)$ and $h(s_3) - h(s_4)$, corresponding to two separate intervals (s_1, s_2) and (s_3, s_4), are independent. The random function

$$\underline{X}(t) = \int_0^t (t - s)dh(s) \tag{5-15}$$

is now considered.

It is easily shown[2] that $\underline{X}(t)$ obeys a Laplace-Gauss law having for typical difference $S(t) = \frac{t^{3/2}}{\sqrt{3}}$ and that, if $\tau = t - t_0$

$$r(t, t_0) = \frac{1 + \frac{3}{2}\frac{\tau}{t}}{\left(1 + \frac{\tau}{t}\right)^{3/2}} = 1 - \frac{3}{8}\frac{\tau^2}{t^2} + \ldots \tag{5-16}$$

Without going into details, it is simply recalled that the demonstration utilizes as intermediary the characteristic function of $\underline{X}(t)$, of which the logarithm, owing to the properties of independence of the $dh(s)$, is expressed by an elementary integral.

From these formulas, it follows that

$$r' = 0, \quad \frac{S'}{S} = \frac{3}{2}\frac{1}{t}$$

and that $R(x, t)$ verifies the first-order partial differential equation

$$\frac{\partial R}{\partial t} + \frac{3}{2}\frac{1}{t}\frac{\partial}{\partial x}(xR) = 0 \tag{5-17}$$

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[2]See, for instance, Bass, reference 4, or "The random functions and their mechanical interpretation." Revue Scientifique, No. 3240, 1945.

$\underline{X}(t)$ is a differentiable random function. Its stochastic derivative is deduced from the expression (5-15) by operations of classical form and written as $\int_0^t dh(s)$. It is easily proved that $\frac{3}{2}\frac{x}{t}$ is the related mean of this derivative, if the value x of $\underline{X}$ is fixed. The equation (5-17) has many solutions which are densities of probability. It does not determine the function K.

Next consider the elementary random function

$$\underline{X}(t) = \int_0^t dh(s) \tag{5-18}$$

stochastic derivative of the function which has been studied as the first illustrative example. It is shown that $\underline{X}(t)$ obeys a Laplace-Gauss law having $S(t) = \sqrt{t}$ for typical difference and that, if $t_0 \leq t$:

$$r\left(t, t_0\right) = \sqrt{\frac{t_0}{t}} = 1 - \frac{1}{2}\frac{\tau}{t_0} + \ldots, \quad r' = -\frac{1}{2t} \tag{5-19}$$

Therefore, $R(x, t)$ verifies the second-order partial derivative equation

$$\frac{\partial R}{\partial t} = \frac{1}{2}\frac{\partial^2 R}{\partial x^2} \tag{5-20}$$

This is the equation of heat. The solution which reduces to a given function $R_0(x)$, for $t = t_0$, is given by the classical formula

$$R(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{t - t_0}} e^{-\frac{1}{2\left(t-t_0\right)}(x-x_0)^2} R\left(x_0\right)dx_0$$

which can also be shown by using the Fourier transform of R, that is, its charactistic function.

Thus equation (5-20) defines here the form of the probability of passage $K(x; x_0, t, t_0)$ contrary to what occurred in equation (5-17).

This example presents an unusual peculiarity. It is easily verified that the function

$$K(x; x_0, t, t_0) = \frac{1}{\sqrt{2\pi}\sqrt{t - t_0}} e^{-\frac{1}{2(t-t_0)}(x-x_0)^2} \tag{5-21}$$

satisfies the functional equation

$$K(x_2; x_0, t_2, t_0) = \int_{-\infty}^{\infty} K(x_2; x_1, t_2, t_1) K(x_1; x_0, t_1, t_0) dx_1 \tag{5-22}$$

called the Chapman-Kolmogoroff equation, which characterizes the Markoff processes (or more generally, the "pseudo-markovian" processes), a functional generalization of simple Markoff chains. This equation can be written in operational notation as

$$K_{t_2 t_0} = K_{t_2 t_1} K_{t_1 t_0} \tag{5-23}$$

It expresses that the operations K form a group. The operator $\frac{K_{t+\tau t'} K_{t' t} - 1}{\tau}$ is then written simply as $\frac{K_{t+\tau t} - 1}{\tau}$. It is independent of t'. In the general case it is its limit L only when $\tau \to 0$ which must be independent of t, but it itself is not.

Returning to equations (5-12) and (5-13) it now is assumed that R is not dependent on t; S is then a constant and $S' = 0$. If $r' \neq 0$, equation (5-13) is written as

$$\frac{\partial}{\partial x}(xR) + S^2 \frac{\partial^2 R}{\partial x^2} = 0 \tag{5-24}$$

The only solution of this differential equation which defines a law of probability is the density of the Laplace-Gauss law

$$R(x) = \frac{1}{S\sqrt{2\pi}} e^{-\frac{x^2}{2s^2}} \tag{5-25}$$

This example contains, as a special case, stationary random functions for which the function $r(t, t_0)$ depends solely on the difference $t - t_0$; r' is then a constant, independent of t.

The final example deals with a vectorial random function of a type to be utilized later. Consider simultaneously the random function

$$X(t) = \int_0^t (t - s)dh(s) \tag{5-26}$$

and its derivative

$$X'(t) = \int_0^t dh(s) \tag{5-27}$$

which can play the part of the speed.

It is easy to form the functions R and K for the vector $\underline{X}$ having X and X' as components. To find the partial differential equation verified by the function $R(x, x', t)$ it is not necessary to first form the kernel K. It is simpler to begin with the expressions of X and X', which, passing through the intermediary of the characteristic function of X, X', gives

$$\frac{\partial R}{\partial t} + x' \frac{\partial R}{\partial x} = \frac{1}{2} \frac{\partial^2 R}{\partial x'^2} \tag{5-28}$$

The operator L is thus a differential operator, of the second order with respect to x':

$$L = -x' \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x'^2} \tag{5-29}$$

Returning now to the old notations and adding the letter u to the velocity vector, we get

$$L = -u \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial u^2} \tag{5-30}$$

considering X'(t) as the velocity of point X(t). The operator L, which is the subject of this example, appertains to the particular class of operators which make it possible to define statistical kinematics and consequently hydrodynamics, as will be proved.

6. Random velocity. Hydrodynamic equations (refs. 3, 4):

The problem involves the separation of what is position and what is velocity in the vectorial random function with six dimensions $\underline{X}(t)$. In classical mechanics the velocity U(t) is the derivative of the position X(t). In the present case, the assumption is made that the velocity is the stochastic derivative of the mean square of the position. By theory of random functions, it follows that, if the random function X(t) is completely or at least known locally, the function U(t) can be defined by operations comparable to those used in classical mechanics to deduce the velocity from the position.

The preceding paragraph contained an example (equations (26) and (27)) of two random functions X(t) and X'(t) = U(t) linked by a relation of this kind, proving in a general way that there is an operator that satisfies the necessary relation:

$$\int ARdu = 0 \tag{6-1}$$

and is such that the basic equation (5-6) (with original notations) takes the form

$$\frac{\partial R}{\partial t} + \sum_k u_k \frac{\partial R}{\partial x_k} = AR \tag{6-2}$$

If A is a differential operator, it can be expressed by an expansion in series (infinite or finite) in terms of the symbolical powers of the partial derivatives $\frac{\partial}{\partial u_k}$ of which the coefficients are functions of x and u. The vectorial operator having for components $\frac{\partial}{\partial x_k}$ occurs only in the first member, by its scalar product with the velocity.

The product $G = RK$ can then also be decomposed in $G = \rho pH$ (compare formulas (4-3) and (5-1)). The function H defines the correlations in the velocity field and function p is the mathematical representation of turbulent diffusion.

Equation (6-2) must therefore play the fundamental part in the theory of turbulence. No attempt is made here at particularization; it simply is shown that it contains the general equations of hydrodynamics.

First, the two members of (6-2) are integrated with respect to u_k, with due regard to (6-1). We introduce the density of the fluid

$$\rho(x, t) = \int R(x, u; t)du \tag{6-3}$$

(or more accurately, a quantity ρ normalized and numerically proportional to the density) and the relative mean of the velocity for a given position, the components of which are

$$\overline{u_i} = \frac{1}{\rho}\int u_i R(x, u; t)du \tag{6-4}$$

We end with an equation of continuity

$$\frac{\partial \rho}{\partial t} + \sum_k \frac{\partial}{\partial x_k}\left(\rho \overline{u_k}\right) = 0 \tag{6-5}$$

which simply expresses the fact that the velocity is the (stochastic) derivative of the position. Thus it is seen that the velocity of the whole of the turbulent motion, defined qualitatively in paragraph 4, has the first of the qualitative properties of the hydrodynamic velocity in general. At the chosen turbulent scale (which is arbitrary) the mass is conserved.

Now it will be seen that the velocity also satisfies the equations of motion. Multiplying equation (6-2) by the velocity component u_i and integrating with respect to u, gives for the first member

$$\frac{\partial}{\partial t}\overline{\rho u_i} + \sum_k \frac{\partial}{\partial x_k}\overline{\rho u_i u_k}$$

Introducing the speed of fluctuation

$$u'_i = u_i - \overline{u_i}$$

and considering the equation of continuity, we put

$$T_{ik} = -\overline{\rho u'_i u'_k} \tag{6-6}$$

The first member becomes

$$\frac{\partial \overline{u_i}}{\partial t} + \sum_k \overline{u_k}\frac{\partial \overline{u_i}}{\partial x_k} - \frac{1}{\rho}\sum_k \frac{\partial T_{ik}}{\partial x_k}$$

the second member is written

$$\int u_i AR \, du = \rho\gamma_i$$

γ_i being a certain function of x which is deduced from the operator A and the probability density R. Hence the equations of motion

$$\frac{\partial \overline{u_i}}{\partial t} + \sum_k \overline{u_k} \frac{\partial \overline{u_i}}{\partial x_k} = \gamma_i + \frac{1}{\rho} \sum_k \frac{\partial T_{ik}}{\partial x_k} \tag{6-7}$$

The γ_i play the part of the components of the density of the field of external forces (gravity, for example). The T_{ik} are the components of the stress tensor (with fixed turbulent scale)3. They are, except for the factor $-\rho$, the components of the correlation tensors of the velocity components at a point. They can be measured directly by statistical methods, and the equations of hydrodynamics for a given turbulent scale can thus be verified.

It should be noted that in this case the equations of motion of the ensemble (average) are involved. For the present, nothing about the behavior of the rate of fluctuation has been assumed. Later on the usual assumption will be made that the velocity satisfies the hydrodynamic equations and, more precisely, the Navier equations. It should be remembered that, based upon this hypothesis, Reynolds was able to establish equations similar to equations (6-7) for the mean turbulent motion. It is apparent that, without it being necessary to repeat Reynolds' calculations, the statistical theory in question here is entirely different from that of Reynolds. It is probably more complete, but it still has not been pushed far enough to be verified by experiment, due to a lack of suitable hypothesis.

As simple example of the fundamental equation (6-2), the case is chosen in which the function $U(t)$ is itself differentiable (in mean squares, refs. 15, 16). In this case, the statistical image point of the fluid has an instantaneous acceleration. If $\Gamma_k(x, u, t)$ represents the relevant mean of the acceleration, that is, of the velocity derivative, when the position and the velocity are fixed, it proves that the density of probability of $X(t)$ and $U(t)$ satisfies the equation

$$\frac{\partial R}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial R}{\partial x_k} + \sum_{k=1}^{3} \frac{\partial}{\partial u_k}(\Gamma_k R) = 0 \tag{6-8}$$

The operator $A = -\sum_{k=1}^{3} \frac{\partial}{\partial u_k}(\Gamma_k R)$ is therefore linear, differential and of the first order. The field of the external forces to which the fluid is subjected has, necessarily, for components, the quantities

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3This expression of stresses was originally given by Reynolds (On the dynamical of incompressible viscous fluids and the determination of the criterion. Phil. trans. Roy. Soc. CLXXXVI, part I, 123, 1895), proceeding from the Navier equations. However, the exact meaning of the Reynolds stresses is different from that of T_{ik}.

$$\gamma_i = -\frac{1}{\rho}\int u_i \sum_{k=1}^{3} \frac{\partial}{\partial u_k}\left(\Gamma_k R\right) du = \frac{1}{\rho}\int \Gamma_i R \, du = \overline{\Gamma}_i \qquad (6\text{-}9)$$

averages formed of the acceleration components when the position is fixed, provided only that $u_i \Gamma_i R$ tends toward zero when the velocity increases to infinity. If, in particular, the Γ_i are not dependent on u, $\overline{\Gamma}_i$ is identical with γ_i, and R satisfies the simple equation

$$\frac{\partial R}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial R}{\partial x_k} + \sum_{k=1}^{3} \gamma_k \frac{\partial R}{\partial u_k} = 0 \qquad (6\text{-}10)$$

Once the γ_k are given, this equation defines $R(x, u, t)$ from $R(x, u, t_0)$. But the form of the linear transformation from $R(x, u, t_0)$ to $R(x, u, t)$, that is, the probability of transition, depends upon the form of the original probability $R(x, u, t_0)$, contrary to what happens, say for the equation of heat (type (5-19)); this illustration does not appear to rest on hypotheses sufficiently inspired by reality to serve as basis of a turbulence theory. First of all, the starting point must be modified, as will be done in the following paragraph:

7. Systems of molecules (refs. 13 and 14):

The statistical quantities to which the analysis of velocity records leads, represent only certain scales of turbulence, those which correspond to the ensemble of "vortices" whose dimensions are superior to a limit approximately fixed by the employed anemometer. Can the theory be changed so that all the possible scales can be represented simultaneously? It seems that it suffices for this purpose to start from the finest scale, that is, the molecular scale. The gas is therefore considered as a system of N molecules, and, to explain the method with as much simplicity as possible, the assumption, which is not verified for air, is made, that the molecules are identical, monatomic, comparable to material points subjected to central interactions. $V(r)$ denotes the potential of the force of interaction of two molecules separated by the distance r. If r is great (with respect to the diameter of the molecules, which will not be introduced explicitly), $V(r)$ is negligible. For the small values of r, $V(r)$ expresses the repulsion of the molecules, generalized form of shocks. No other information about $V(r)$ is needed beforehand, at least in a general theory.

The motion of this system of molecules is controlled by the equations of dynamics. But the extreme complication of the trajectories of the

molecules prompted the replacement of rational mechanics by statistical mechanics. We prefer to introduce random mechanics where each molecule is a random point. By means of an ergodic hypothesis the random motion of a molecule can be considered as being statistically equivalent to the motion of the ensemble of the fluid, in quasi-steady conditions, the ensemble of successive states of the visualized molecule replacing the ensemble of the simultaneous states of all the molecules. The molecule would therefore be the concrete image of the abstract random point serving up to now for representing the fluid. The molecular scale is a true ultimate scale of turbulence, separated, however, from the actual turbulent scales by a poorly defined but finite interval. While for experimental turbulence the concept of a random image point is a mathematical abstraction, it is a natural idea and a starting point for the molecules.

This idea is now explored but by a method slightly different from that discussed in the preceding paragraphs. The probability density $f_N(x_1, \ldots, x_N, u_1, \ldots, u_N; t)$ of the positions and the velocities of N molecules simultaneously, rather than singly, is introduced. The gas appears then as a random point with three N dimensions, in a space of configuration, and no longer as a random point of ordinary and physical space. For the time being, the notations $x_1, x_2, \ldots, x_N$ shall have a vectorial character and represent the system of the three coordinates of the molecules[4] numbered 1, 2, . . ., N.

The potential of interaction of the molecules of rank i and j is indicated by V_{ij}, and the external force to which the molecule of rank i is subjected, by γ_i. If m is the mass of a molecule, the equations of motion of the molecules read

$$\frac{dx_i}{dt} = u_i \qquad m\frac{du_i}{dt} = \gamma_i + \sum \frac{\partial V_{ij}}{\partial x_j} \tag{7-1}$$

the summation applying to all values of j from 1 to N, when V_{ii} is assumed to be zero. Naturally $V_{ij} = V_{ji}$.

When these equations are written, the hypothesis is made that the velocity of each molecule is differentiable. It follows, that the random

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[4]In the preceding paragraphs the notation $f(x, t)$ represented already the probability density of a vector x, of components x_1, x_2, x_3. What is used only temporarily is the meaning of the subscripts, particular to this paragraph. A change in notation could be avoided only by complications in writing which would be more harmful than useful.

point of 3N dimensions having for coordinates the ensemble of the coordinates of the molecules is represented by a random vector function of the configuration space of 3N dimensions doubly differentiable. This property which, as stated before, is not necessarily true for the three-dimensional random point used previously, is the reason, according to the theory of statistical functions, why f_N verifies the partial derivative equation

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} u_i \frac{\partial f_N}{\partial x_i} + \frac{1}{m} \sum_{i=1}^{N} \left(\gamma_i + \sum_{j=1}^{N} \frac{\partial V_{ij}}{\partial x_j} \right) \frac{\partial f_N}{\partial u_i} = 0 \qquad (7\text{-}2)$$

which expresses that f_N is an integral of the equations of motion. It has the same form as Liouville's equation of statistical mechanics. But it should be remembered that its original meaning is a little different. This equation replaces and defines, in the space of 3N dimensions, the equation (6-2), and must now be exploited.

The turbulence involves "particles" or "eddies" formed, at a given scale, by groups of s molecules, s being a very large number, but at the same time very small with respect to N. Let us investigate what functional equation is satisfied by the probability density f_s relative to the molecules of rank 1, 2, . . ., s. f_s is defined by

$$f_s = \int f_N \, dx_{s+1} \, du_{s+1} \cdots dx_N \, du_N \qquad (7\text{-}3)$$

The fundamental equation (7-2) must be integrated with respect to x_{s+1}, u_{s+1}, . . ., x_N, u_N. The operation is obvious, except for the terms in $\frac{\partial V_{ij}}{\partial x_j}$; V_{ij} being function solely of u_i and x_j, the integration gives a zero result if i and j are both superior to s.

If i and j are both inferior to s, then $\frac{1}{m} \frac{\partial V_{ij}}{\partial x_j} \frac{\partial f_s}{\partial u_i}$.

If $j \leq s$, $i > s$, the integration with respect to u_i shows that the result still is zero.

If, finally, $i \leq s$, $j > s$, we first can integrate with respect to x_k, u_k, for $k \neq i$.

The result is $\frac{1}{m}\frac{\partial V_{ij}}{\partial x_j}\frac{\partial f_{s+1}}{\partial u_i}$, where f_{s+1} represents the probability density relative to $s + 1$ molecules of rank 1, 2, . . ., s, j. The last integration

$$\frac{1}{m}\int\frac{\partial V_{ij}}{\partial x_j}\frac{\partial f_{s+1}}{\partial u_i}\,dx_j\,du_j = \frac{1}{m}\frac{\partial}{\partial u_i}\int\frac{\partial V_{ij}}{\partial x_j}f_{s+1}\,dx_j\,du_j$$

cannot be extended farther, because both $\frac{\partial V_{ij}}{\partial x_j}$ and f_{s+1} are functions of x_j.

Now it will be noted that in the sum

$$\frac{1}{m}\sum_{j=s+1}^{N}\frac{\partial}{\partial u_i}\int\frac{\partial V_{ij}}{\partial x_j}f_{s+1}\,dx_j\,du_j$$

the terms are identical because the molecule j plays an anonymous part.

The final result is

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} u_i\frac{\partial f_s}{\partial x_i} + \frac{1}{m}\sum_{i=1}^{s}\gamma_i\frac{\partial f_s}{\partial u_i} = \frac{N - s}{m}\sum_{i=1}^{s}\frac{\partial}{\partial u_i}\int\frac{\partial V_{is+1}}{\partial x_{s+1}}f_{s+1}\,dx_{s+1}\,du_{s+1} \tag{7-4}$$

This equation seems capable of serving as basis for a theory of turbulence. The scale there appears explicitly for the numbers $\frac{1}{s}$ and $\frac{s}{N}$ and the problem (not taken up here) consists in formulating a reasonable hypothesis which enables f_{s+1} to be expressed with the aid of f_s, so that (7-4) becomes a functional equation in f_s.

For $s = 1$, (7-4) is, in a certain measure, comparable to (6-2) and reads

$$\frac{\partial f_1}{\partial t} + u_1 \frac{\partial f_1}{\partial x_1} + \frac{1}{m}\gamma_1 \frac{\partial f_1}{\partial u_1} = \frac{N - 1}{m} \frac{\partial}{\partial u_1} \int \frac{\partial V_{12}}{\partial x_2} f_2 \, dx_2 \, du_2 \qquad (7\text{-}5)$$

Equation (7-5) differs from (6-2) by the presence of the terms in f_2. The presence of the second member relates (7-5) to (6-2), but its form makes it different from it. Equation (7-5) reduces to a functional equation in f_1 only if f_2 is tied to f_1 by a suitable assumption. Such an assumption has been made by J. Yvon who demonstrated with its aid Boltzmann's fundamental equation on which the kinetic theory of gases is based. More recently it was taken up again by Born and Green, who also studied the case of $s = 2$, and applied it to the case of liquids on the basis of an assumption by Kirkwood binding f_3 to f_2.

It seems that the case of large values of s has never been studied.

Equation (7-5) like (6-2) and despite the not necessarily linear character of the second member with respect to f_s, involves equations of hydrodynamics which are easy to write, if the method indicated in paragraph 6 is applied. Since these equations have so far not been used in concrete applications to turbulence, only these summary indications are given here. The application of equation (7-4) to turbulence raises difficulties which are far from being solved, or even stated but it is important to know that this equation exists.

CHAPTER II

CORRELATIONS AND SPECTRAL FUNCTIONS

8. Introduction - correlations in space - homogeneity, isotropy:

In this chapter we deal no longer with the general laws of probability of turbulence. We limit ourselves to the study of the correlations of the law of the field $H(u,u'; x,x',t,t')$. Following are some preliminary remarks on the subject:

Experiments furnish the correlations in form of temporal averages. Chapter I shows how it was possible to change their interpretation for converting them into stochastic averages. The assumption is made that

this operation is always possible. But for starting, there is no necessity for knowing whether it has been effected, since only properties of symmetry, of tensional character, are involved here. On the other hand, arriving at the dynamics of turbulence we shall see that the use of temporal averages leads to serious difficulties, which disappear when they are transformed into stochastic averages, and our calculations will deal only with stochastic averages. Temporal averages are resorted to only when the calculations have reached the laws which must be compared with physical reality.

The components of the speed fluctuations[5] at point x_1,x_2,x_3 are designated by u_1, u_2, and u_3. They are statistical quantities whose mean value is zero. Time plays no part at present and the parameter t in the formulas is disregarded. The correlations of the velocity between the two points x and $x' = x + \xi$ are defined by a tensor having for components

$$R_{\alpha\beta}(x,x') = \overline{u_\alpha(x)u_\beta(x')} \tag{8-1}$$

The scalar is introduced also:

$$R = \sum_\alpha R_{\alpha\alpha} = \sum \overline{u_\alpha(x)u_\alpha(x')} \tag{8-2}$$

The turbulence is said to be homogeneous (within a certain field of space) when the $R_{\alpha\beta}$ are not separately dependent on the two points x,x', but only on their relative position, that is, on the vector $\xi = x' - x$. Then $R_{\alpha\beta}(\xi)$ replaces $R_{\alpha\beta}(x,x')$.

The turbulence is said to be isotropic at point x if it is homogeneous in the vicinity of x and the tensor $R_{\alpha\beta}$ is invariant to any rotation of the axes and any reflection. The form which isotropy imposes on the components of this tensor is discussed later.

These two definitions concern only the second-rank tensor $R_{\alpha\beta}$. They are "second-rank properties" of the statistical vector u_α. They must be extended to certain third-rank tensors. Homogeneity and isotropy can be given a more complete and also more restrictive definition by extension to all possible moments of velocity components taken at any number of points. It is then more convenient to define the homogeneity

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[5]They were called u'_1,u'_2,u'_3 in chapter I. But it is advantageous to use thereafter the notation u' for other purposes.

as the invariance to translations of the law of probability of the ensemble of velocity vectors having for origin a certain number of arbitrarily distributed points, and the isotropy, once homogeneity is achieved, as invariance to rotations and reflections.

Without restrictive hypotheses the tensor $R_{\alpha\beta}$ has nine distinct components, functions of the seven variables $x_1, x_2, x_3, x'_1, x'_2, x'_3, t$. If the turbulence is homogeneous, the nine functions remain, but of four variables only ξ_1, ξ_2, ξ_3, t. The isotropy is now written.

To the tensor $R_{\alpha\beta}(\xi)$ we associate the scalar bilinear form

$$\sum R_{\alpha\beta} X_\alpha Y_\beta$$

where X_α and Y_α are two arbitrary vectors. This form is the mean value of the product of two scalar products $\sum u_\alpha(x) X_\alpha$ and $\sum u_\alpha(x') Y_\alpha$. If it is invariant to rotations, it is an algebraic combimation, separately linear with respect to X_α and Y_α, of invariants of vectors $X_\alpha, Y_\alpha, \xi_\alpha$ with respect to rotations. These invariants are

$$\sum X_\alpha Y_\alpha \qquad \sum \xi_\alpha X_\alpha \qquad \sum \xi_\alpha Y_\alpha \qquad r^2 = \sum \xi_\alpha^2$$

There are therefore two scalars $A(r)$, $B(r)$, functions of r, so that

$$\sum R_{\alpha\beta} X_\alpha Y_\beta = A \sum X_\alpha Y_\alpha + B \left(\sum \xi_\alpha X_\alpha \right) \left(\sum \xi_\beta Y_\beta \right) \qquad (8\text{-}3)$$

The identification shows that, if $\delta_{\alpha\beta}$ represents the classical symbol of Kronecker, zero if $\beta \neq \alpha$, equal to unity when $\beta = \alpha$:

$$R_{\alpha\beta} = A\delta_{\alpha\beta} + B\xi_\alpha \xi_\beta \qquad (8\text{-}4)$$

The tensor $R_{\alpha\beta}$, in the isotropy hypothesis, depends therefore solely on two distinct functions of two variables r and t. The notations are changed as usual and one puts with Karman and Howarth (reference 30):

$$R_{\alpha\beta} = u_0^2\left[\frac{f - g}{r^2}\xi_\alpha\xi_\beta + g\delta_{\alpha\beta}\right] \tag{8-5}$$

f and g being functions of r, t becoming unity for $r = 0$, and u_0^2 a simple function of t.

If $r = 0$, $R_{\alpha\beta}$ is reduced to

$$R_{\alpha\beta}(0) = u_0^2 g(0)\delta_{\alpha\beta} \tag{8-6}$$

It is seen that

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = u_0^2 \qquad \overline{u_2u_3} = \overline{u_3u_1} = \overline{u_1u_2} = 0 \tag{8-7}$$

Also, owing to the isotropy, the tensor $R_{\alpha\beta}$ is symmetrical.

Lastly, it should be noted that the quadratic form $\sum R_{\alpha\beta}X_\alpha X_\beta$ is the first member of the equation of an ellipsoid of revolution. This shows that, if one takes, no matter how, a symmetrical table of numbers $R_{\alpha\beta}$, they are not, in general, components of a correlation tensor. These numbers must verify the inequalities which state that they are coefficients of the first member of the equation of an ellipsoid. In other words, the roots of the equation of the third degree $\left|R_{\alpha\beta} - S\delta_{\alpha\beta}\right| = 0$ (equation in S classical) must be positive.

A similar method permits the reducing of the components of the tensor

$$T_{\alpha\beta\gamma} = \overline{u_\alpha(x)u_\beta(x)u_\gamma(x + \xi)} \tag{8-8}$$

which will be needed later. This tensor is symmetrical with respect to α and β, and the trilinear form

$$\sum T_{\alpha\beta\gamma} X_\alpha Y_\beta Z_\gamma$$

is invariant to rotations. By applying the classical notations of Kármán and Howarth, it is found, after a few calculations, that

$$T_{\alpha\beta\gamma} = u_0^3 \left[\frac{a}{r} \xi_\alpha \delta_{\beta\gamma} + \frac{a}{r} \xi_\beta \delta_{\gamma\alpha} + \frac{b}{r} \xi_\gamma \delta_{\alpha\beta} + \frac{c - b - 2a}{r^3} \xi_\alpha \xi_\beta \xi_\gamma \right] \quad (8\text{-}9)$$

a, b, and c being three functions of r and t.

9. Properties of the functions f, g, a, b, c. Incompressibility[6]:

Other forms of symmetry less particular than isotropy can be visualized, as for instance, axial symmetry, or invariance to rotations about a given axis, instead of about a point. The case of isotropy is that in which the $R_{\alpha\beta}$ depend on the smallest number of separate functions.

The functions f and g are correlation coefficients. For example, f can be defined by taking two velocity components along the axis of x_1 (direction of velocity of the ensemble) at two points situated on a parallel to this axis (fig. 3)

$$f(r) = \frac{\overline{u_1(x_1,x_2,x_3)\,u_1(x_1 + r,x_2,x_3)}}{u_0^2} \quad (9\text{-}1)$$

g is defined by taking two components still parallel to the axis of the x_1, but at two points located on a line perpendicular to this axis.

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[6]The authors (reference 30) use q, l, k where we use a, b, c. But the letter k is also used in a just as classical although more recent fashion to designate the spectral frequency and will be used with this meaning in the present report.

For example

$$g(r) = \frac{\overline{u_1(x_1,x_2,x_3)u_1(x_1,x_2 + r,x_3)}}{u_0^2} \tag{9-2}$$

Experience indicates that near $r = 0$, the functions f and g are continuous and twice differentiable and allow a tangent to the "horizontal" origin (fig. 4).

Theory confirms experience, which unfortunately lacks precision when the distance r becomes small. Much greater accuracy is obtained by correlation measurements with difference in time (experiments by Favre, Report to the VIIth Intern. Cong. of Appl. Mech., London, 1948), because only one anemometer is used and the timing can be reduced as much as desired.

Hence, one may write developments of the form

$$\left.\begin{aligned} f(r) &= 1 + \frac{r^2}{2}f''(0) + \ldots \\ g(r) &= 1 + \frac{r^2}{2}g''(0) + \ldots \end{aligned}\right\} \tag{9-3}$$

$f''(0)$ and $g''(0)$ are negative quantities, possibly time functions, having the dimensions of the inverse of the square of a length. The length

$$\lambda = \sqrt{-\frac{2}{g''(0)}} \tag{9-4}$$

is called length of dissipation.

The triple-correlation functions a,b,c have interpretations similar to those of f and g. For example:

$$c(r) = \frac{1}{u_0^3}\overline{u_1^2(x_1,x_2,x_3)u_1(x_1 + r,x_2,x_3)} \tag{9-5}$$

It follows from formula (8-9) that, if $T_{\alpha\beta\gamma}$ has a well defined limit when $\xi \longrightarrow 0$, this limit can only be zero. Hence, $a(0), b(0), c(0)$ are zero. In a general way, $T_{\alpha\beta\gamma}$ is an odd function of $\xi_\alpha, \xi_\beta, \xi_\gamma$. Consequently, considered as function of r, $c(r)$ is an odd function. Its development has no term in r^2. Lastly, it is shown that it has no term in r, either. The coefficient of this term is the mean of

$$u_1^2 \frac{\partial u_1}{\partial x_1} = \frac{1}{3} \frac{\partial}{\partial x_1} u_1^3 = \frac{1}{3} \lim_{r \longrightarrow 0} \frac{u_1^3(x_1 + r, x_2, x_3) - u_1^3(x_1, x_2, x_3)}{r}$$

or

$$\frac{1}{3} \lim_{r \longrightarrow 0} \frac{1}{r} \left[\overline{u_1^3(x_1 + r_1, x_2, x_3)} - \overline{u_1^3(x_1, x_2, x_3)} \right]$$

But on account of the homogeneity extended to the averages of the third order, $\overline{u_1^3(x_1 + r, x_2, x_3)}$ is equal to $\overline{u_1^3(x_1, x_2, x_3)}$. Hence the limit is zero. Lastly, $c(r)$ is an infinitesimal of the third order with respect to r. The same holds for $b(r)$ and $a(r)$.

If the flow is incompressible, the various functions that characterize the correlations of isotropic turbulence are not independent. From the relation $\sum \frac{\partial u_\alpha}{\partial x_\alpha} = 0$ it is, in fact, immediately deduced that

$$\sum_\alpha \frac{\partial R_{\alpha\beta}}{\partial x_\alpha} = 0 \qquad (9\text{-}6)$$

This equation is general. In the particular case of isotropic turbulence it leads to Kármán's equation

$$g = f + \frac{r}{2} \frac{\partial f}{\partial r} \qquad (9\text{-}7)$$

In the same manner, one then finds that:

$$c = -2b \qquad a = -b - \frac{r}{2}\frac{\partial b}{\partial r} \tag{9-8}$$

So, of the five functions f,g,a,b,c which define the double and triple correlations of isotropic turbulence, only two remain distinct if the flow is incompressible.

The incompressibility results, in particular, in the relation $g''(0) = 2f''(0)$. Therefore, in terms of dissipation length λ:

$$\left.\begin{aligned} f(r) &= 1 - \frac{r^2}{2\lambda^2} + \ldots \\ g(r) &= 1 - \frac{r^2}{\lambda^2} + \ldots \end{aligned}\right\} \tag{9-9}$$

Together with the length λ, two other numerical parameters (time functions) are frequently used to give an idea of the turbulence.

First, the correlation length

$$L = \int_0^\infty f(r)\, dr \tag{9-10}$$

is introduced; it depends upon all values of $f(r)$ for $0 < r < \infty$, while λ depends only on the form of the function $f(r)$ at the origin. It may be pointed out that L is expressed by means of the scalar $R = \sum \overline{u_\alpha(x)u_\alpha(x + \xi)} = u_0^2(f + 2g)$. Owing to the incompressibility, $f + 2g = 3f + r\frac{\partial f}{\partial r} = 2f + \frac{\partial}{\partial r}(rf)$. If $rf(r)$ approaches zero when $r \longrightarrow \infty$:

$$\int_0^\infty (f + 2g)\, dr = 2L \tag{9-11}$$

The intensity of turbulence in one direction, that of the axis of x_1, for example, is the dimensionless quantity

$$\frac{\sqrt{\overline{u_1^2}}}{U} \tag{9-12}$$

where U is the mean velocity. The total intensity of turbulence is the ratio

$$\frac{\frac{1}{3}\sqrt{\sum \overline{u_\alpha^2}}}{U} = \frac{1}{3}\frac{\sqrt{R(0)}}{U} \tag{9-13}$$

If the turbulence is isotropic, its intensity is the same in every direction. It is equal to the total intensity and to $\frac{u_0}{U}$.

In the case of isotropic turbulence the mean square values of the rotational components are directly associated with the quantities u_0 and λ.

Introducing the components

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \qquad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \qquad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

of the rotational, the averages $\overline{\omega_1^2}$, $\overline{\omega_2^2}$, $\overline{\omega_3^2}$ for homogeneous and and isotropic turbulence are computed:

$$\overline{\omega_1^2} = \overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2} + \overline{\left(\frac{\partial u_2}{\partial x_3}\right)^2} - 2\overline{\frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}}$$

We note that, for example:

$$\frac{\partial u_3}{\partial x_2} = \lim_{\xi_2 \to 0} \frac{u_3(x_1, x_2 + \xi_2, x_3) - u_3(x_1, x_2, x_3)}{\xi_2}$$

$$\overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2} = \lim_{\xi_2 \to 0} \frac{1}{\xi_2^2}\left[\overline{u_3^2(x_1, x_2 + \xi_2, x_3)} + \overline{u_3^2(x_1, x_2, x_3)} - \right.$$

$$\left. \overline{2u_3(x_1, x_2, x_3)u_3(x_1, x_2 + \xi_2, x_3)}\right]$$

The first two bracketed terms are equal to u_0^2. The third is equal to $R_{33}(0,\xi_2,0) = u_0^2 g$, or, except for the third order, $u_0^2\left(1 - \frac{\xi_2^2}{\lambda^2}\right)$. Lastly

$$\overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2} = \frac{2u_0^2}{\lambda_2}$$

Likewise

$$\overline{\left(\frac{\partial u_2}{\partial x_3}\right)^2} = \frac{2u_0^2}{\lambda_2}$$

Lastly:

$$\overline{\frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}} = \lim_{\xi_2,\xi_3 \to 0} \overline{\frac{u_3(x_1,x_2+\xi_2,x_3) - u_1(x_1,x_2,x_3)}{\xi_2} \frac{u_2(x_1,x_2,x_3+\xi_3) - u_2(x_1,x_2,x_3)}{\xi_3}}$$

$$= \lim \frac{1}{\xi_2\xi_3}\left[R_{23}(0,\xi_2,-\xi_3) - R_{12}(0,0,\xi_3) - R_{23}(0,\xi_2,0) + R_{12}(0,0,0)\right]$$

$$= u_0^2 \lim \frac{1}{\xi_2\xi_3}\left[-\frac{f-g}{r^2}\xi_2\xi_3\right] = -\frac{u_0^2}{2\lambda_2}$$

Finally

$$\overline{\omega_1^2} = 2\frac{u_0^2}{\lambda^2} + 2\frac{u_0^2}{\lambda^2} + \frac{u_0^2}{\lambda^2} = 5\frac{u_0^2}{\lambda^2}$$

Naturally, $\overline{\omega_2^2}$ and $\overline{\omega_3^2}$ are equal to $\overline{\omega_1^2}$.

10. Spectral decomposition of the velocity:

We know that the representative curve of a turbulent velocity component $u(t)$ in terms of time suggests the idea of an irregular periodic phenomenon. The analytical representation of $u(t)$ is not a periodic Fourier series, but rather an almost periodic series or a Fourier integral.

In the first case, $u(t)$ is a sum of harmonics without common base period; by adopting the complex notation

$$u(t) = \sum_{n=-\infty}^{+\infty} A_n e^{i\omega_n t} \qquad (10\text{-}1)$$

the pulsations ω_n forming a succession of real numbers increasing with n. It can always be supposed that $\omega_{-n} = -\omega_n$. $u(t)$ is a real quantity if $A_{-n} = A_n^*$, the notation A_n^* designating the conjugate imaginary of A_n. The series $|A_n|$ must be convergent.

In the second case:

$$u(t) = \int_{-\infty}^{\infty} A(\omega) e^{i\omega t} d\omega \tag{10-2}$$

the function $A(\omega)$ being absolutely summable and such that $A(-\omega) = A^*(\omega)$.

The numbers A_n or the function $A(\omega)$ depend upon the position of the anemometer.

This representation is practical when the turbulence is steady in time, which enables a vigorous application of the ergodic principle and the calculation of the averages connected with $u(t)$ in a time interval as great as desired. But, among the simplest problems involved, there is, first of all, that of a turbulence homogeneous in space (at least in a sufficiently restricted range) and developing in time. The pseudo-periodic character of the speed is then manifested in space rather than in time. (A detailed study follows.)

The component u_α of the velocity of fluctuation at point x_1, x_2, x_3 at a given instant can be developed in series or by Fourier integral, depending upon whether the spectrum is discrete or continuous

$$u_\alpha(x,t) = \sum_{n_1, n_2, n_3} Z_\alpha^{(n_1,n_2,n_3)}(t) e^{i\left(x_1\lambda_1^{(n_1)} + x_2\lambda_2^{(n_2)} + x_3\lambda_3^{(n_3)}\right)} \tag{10-3}$$

$$u_\alpha(x,t) = \int_\Lambda Z_\alpha(\lambda_1,\lambda_2,\lambda_3,t) e^{i(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)} d\lambda_1\, d\lambda_2\, d\lambda_3 \tag{10-4}$$

n_1, n_2, and n_3 are three integers that vary independently from $-\infty$ to $+\infty$ in formula (10-3), and on which the quantities $Z_\alpha^{(n_1,n_2,n_3)}$, $\lambda_1^{(n_1)}$, $\lambda_2^{(n_2)}$, $\lambda_3^{(n_3)}$ are dependent. In formula (10-4), u_α is a triple integral extended over the entire space Λ of wave numbers (The wave number, inverse of a length, is the equivalent in space to the frequency, inverse of time. It is a vector). u_α is a real quantity if $Z_\alpha(-\lambda_1,-\lambda_2,-\lambda_3,t) = Z_\alpha^*(\lambda_1,\lambda_2,\lambda_3,t)$.

These notations are of classical form. But, since the u_α are steady, random functions of space, it is preferable to represent them as stochastic Fourier integrals

$$u_\alpha(x,t) = \int_\Lambda e^{i\left(\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3\right)} dh_\alpha(\lambda,t) \qquad (10\text{-}5)$$

In this formula, the $h_\alpha(\lambda,t)$ are random functions of $\lambda_1,\lambda_2,\lambda_3$ with orthogonal increments (or noncorrelated). Or, in other words, if λ and λ' are two different points in the space Λ, the mean of the product of the two increments $dh_\alpha(\lambda,t)$ and $dh_\beta(\lambda',t)$ is zero. On account of the complex notations it naturally is a question of a product of hermitian symmetry. Hence

$$\overline{dh_\alpha{}^*(\lambda,t)\ dh_\beta(\lambda',t)} = 0 \quad \text{if} \quad \lambda' \neq \lambda \qquad (10\text{-}6)$$

If $\lambda' = \lambda$ at the same point, this average is, in general, no longer zero. It is infinitely small of the order of $d\lambda$. We put

$$\overline{dh_\alpha{}^*(\lambda,t)\ dh_\beta(\lambda,t)} = \varphi_{\alpha\beta}(\lambda,t)\ d\lambda \qquad (10\text{-}7)$$

Now the spectrum of turbulence in the different cases (10-3), (10-4), and (10-5) is defined.

11. Spectral tensor and correlation tensor:

Assuming spatial homogeneity it is now attempted to form, from the spectral decomposition of the velocity, the expression of the components $R_{\alpha\beta}(\xi_1,\xi_2,\xi_3,t)$ or abbreviated, $R_{\alpha\beta}(\xi)$ of the correlation tensor of the velocity at a given instant.

If the formulas (10-3) or (10-4) are utilized, the spatial averages in a very great volume, physically limited but practically infinite with respect to microscopic turbulent lengths, must be used. It involves a cube with edges parallel to the axes, of arbitrarily large length $2a$, of which the center, which can be placed anywhere because of the homogeneity, is placed in the origin of the axes of the coordinates.

First, take the case of the Fourier series (10-3). To form $R_{\alpha\beta}$, involves, first, the product $u_\alpha(x)u_\beta(x + \xi)$, by associating an arbitrary term of the series which represents $u_\alpha(x)$, or rather $u_\alpha^*(x)$, to any one term of the series that represents $u_\beta(x + \xi)$. To form the average in the volume $V = 8a^3$, involves division by V and integration of x_α in space. The exponentials are restricted, and likewise their integrals in V, so that the quotients by V approach zero when $a \longrightarrow \infty$. There is an exception for the terms of the series $u_\alpha(x)$ and $u_\alpha(x + \xi)$ which correspond to the same wave numbers, and which give the products

$$Z_\alpha^*(n_1,n_2,n_3)\, e^{-i \sum_{p=1}^{3} x_p \lambda_p^{(n_p)}} \times Z^{(n_1,n_2,n_3)}\, e^{i \sum_{p=1}^{3} (x_p+\xi_p) \lambda_p^{(n_p)}} =$$

$$Z_\alpha^*(n_1,n_2,n_3)\, Z_\beta(n_1,n_2,n_3)\, e^{i \sum_{p=1}^{3} \xi_p \lambda_p^{(n_p)}} \tag{11-1}$$

independent of x_α. They are therefore equal to their mean value and consequently

$$R_{\alpha\beta}(\xi) = \overline{u_\alpha(x)u_\beta(x + \xi)} =$$

$$\sum_{n_1,n_2,n_3} Z_\alpha^*(n_1,n_2,n_3)\, Z_\beta(n_1,n_2,n_3)\, e^{i \sum_{p=1}^{3} \xi_p \lambda_p^{(n_p)}} \tag{11-2}$$

This is none other than the Parseval formula for the almost periodic Fourier series of three variables. It naturally assumes conditions of uniform convergence which need not be defined here.

In the case of the Fourier integral, $u_\alpha(x)u_\beta(x+\xi)$ is a sextuple integral

$$u_\alpha(x)u_\beta(x+\xi) = \int_{\Lambda\times\Lambda} Z_\alpha^*(\lambda)Z_\beta(\lambda')e^{-i\sum_{p=1}^{3}\lambda_p x_p + i\sum_{p=1}^{3}\lambda'_p(x_p+\xi_p)}\,d\lambda\,d\lambda' \tag{11-3}$$

extended over all values of λ_1, λ_2, λ_3, λ'_1, λ'_2, and λ'_3.

Integrating under the sign $\int$ in the finite volume V and dividing by V, we find that, mathematically speaking, the mean value which is being sought is the limit for an infinite of

$$\frac{1}{V}\int_{\Lambda\times\Lambda} Z_\alpha^*(\lambda)Z_\beta(\lambda')e^{i\sum_{p=1}^{3}\lambda'_p\xi_p}\,d\lambda\,d\lambda' \int_V e^{i\sum_{p=1}^{3}(\lambda'_p-\lambda_p)x_p}\,dx =$$

$$\frac{8}{V}\int_{\Lambda\times\Lambda} Z_\alpha^*(\lambda)Z_\beta(\lambda')e^{i\sum_{p=1}^{3}\lambda'_p\xi_p}\,d\lambda\,d\lambda' \frac{\sin(\lambda'_1-\lambda_1)a}{\lambda'_1-\lambda_1}$$

$$\frac{\sin(\lambda'_2-\lambda_2)a}{\lambda'_2-\lambda_2}\,\frac{\sin(\lambda'_3-\lambda_3)a}{\lambda'_3-\lambda_3} \tag{11-4}$$

Suppose $Z_\alpha(\lambda)$ satisfies "Dirichlet's conditions" or, more generally, is a limited variation; then the integral which figures in (11-4) has a finite limit which, when $Z_\alpha(\lambda)$ is continuous, has for value

$$8\pi^3\int_\Lambda Z_\alpha^*(\lambda)Z_\beta(\lambda)e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\,d\lambda \tag{11-5}$$

This is Dirichlet's theorem.

The mean value of $\overline{u_\alpha(x)u_\beta(x+\xi)}$, the quotient of this integral by $V = 8a^3$, thus approaches zero when $a \longrightarrow \infty$, and the formalism of the Fourier integral does not furnish the spectrum. This conclusion is paradoxical.

But every difficulty disappears when it is assumed that the turbulence might have only a line spectrum, incompatible with the so-called Fourier integral. But such an assumption does not seem reasonable.

It seems more satisfactory to concede that the functions representing the turbulent velocity are too complicated for applying Dirichlet's theorem and, more precisely, that their oscillations are too crowded to be represented by functions with bounded variation. A mathematical process avoiding this difficulty is to follow.

But an approximate argument can also be made. In reality, a is very great (with respect to turbulent wave lengths), but finite. Hence the mean value of $u_\alpha(x)u_\beta(x+\xi)$ is given by formula (11-4), without transition to the limit. However, a transition to the approximate limit can be made and assumed that, with suitable accuracy

$$\overline{u_\alpha(x)u_\beta(x+\xi)} = \frac{8\pi^3}{V}\int_\Lambda Z_\alpha^*(\lambda)Z_\beta(\lambda)e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\,d\lambda \qquad (11\text{-}6)$$

So, if a spectral function $\varphi_{\alpha\beta}(\lambda)$ is defined by

$$R_{\alpha\beta}(\xi) = \overline{u_\alpha(x)u_\beta(x+\xi)} = \int_\Lambda \varphi_{\alpha\beta}(\lambda)e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\,d\lambda \qquad (11\text{-}7)$$

we get

$$\varphi_{\alpha\beta}(\lambda) = \frac{8\pi^3}{V}Z_\alpha^*(\lambda)Z_\beta(\lambda) \qquad (11\text{-}8)$$

In a more accurate way, the $\varphi_{\alpha\beta}(\lambda)$ constitute the components of a tensor, the spectral tensor, transformed from Fourier's correlation tensor. But the approximate formula (11-8) has the drawback of yielding a spectral tensor of a too restrictive form. In fact, there is no reason that the $\varphi_{\alpha\beta}$ be the product of a function of subscript α by a function of subscript β, that is, the general product of the vector $\sqrt{\frac{8\pi^3}{V}}Z_\alpha(\lambda)$ by itself. This new defect can be corrected by superposing a temporal average on the spatial average (11-8). But it is much preferred to turn elsewhere and have recourse to the representation of the velocity by Fourier's stochastic integral (10-5). The product $u_\alpha(x)u_\beta(x+\xi)$ is a double stochastic integral

$$u_\alpha(x)u_\beta(x+\xi) = \int_{\Lambda\times\Lambda} e^{-i\sum_{p=1}^{3}\lambda_p x_p + i\sum_{p=1}^{3}\lambda'_p(x_p+\xi_p)}\, dh_\alpha^*(\lambda)dh_\beta(\lambda') \tag{11-9}$$

On averaging, it is seen that the quantity $\overline{dh_\alpha^*(\lambda)dh_\beta(\lambda')}$ is involved which is, as explained earlier, zero when $\lambda' \neq \lambda$, and of the form $\varphi_{\alpha\beta}(\lambda)d\lambda$ if $\lambda' = \lambda$. Hence simply

$$R_{\alpha\beta}(\xi) = \overline{u_\alpha(x)u_\beta(x+\xi)} = \int_\Lambda e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\, \varphi_{\alpha\beta}(\lambda)d\lambda \tag{11-10}$$

The correlation tensor $R_{\alpha\beta}(\xi)$ is the Fourier transform of the spectral tensor $\varphi_{\alpha\beta}(\lambda)$. The $\varphi_{\alpha\beta}(\lambda)$ here have the generalty desirable. They are subject to the two following conditions:

(a) $\varphi_{\alpha\beta}(\lambda)$ has hermitian symmetry

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^*$$

(b) Whatever the complex numbers X_α, the hermitian form

$$\sum_{\alpha,\beta=1}^{3} X_\alpha X_\beta^* \varphi_{\alpha\beta}$$

which is real according to the first condition, cannot be negative.

Any system of numbers $\varphi_{\alpha\beta}$ constituting a tensor and satisfying these two conditions can, a priori, be used as spectral tensor.

12. Spectral tensor of isotropic, incompressible turbulence:

I. Suppose, with Heisenberg (reference 23), that the turbulence is isotropic. Then, as supplementary condition, the flow is incompressible.

The method used to express the isotropy of the correlation tensor is applied to all tensors, whatever its meaning. The spatial tensor is isotropic when two functions $A(k)$, $B(k)$ exist depending, moreover, on t such that

$$\varphi_{\alpha\beta}(\lambda) = A(k)\lambda_\alpha\lambda_\beta + B(k)\delta_{\alpha\beta} \qquad (12\text{-}1)$$

k^2 denoting the quantity $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

If the flow is incompressible, the functions A and B are formed by a relation equivalent to Kármán's relation (9-7) between the functions of the correlation f and g. We write, in fact, that $\sum \frac{\partial u_\alpha}{\partial x_\alpha} = 0$. Since

$$\sum \frac{\partial u_\alpha}{\partial x_\alpha} = i \int_\Lambda e^{i \sum_{p=1}^{3} \lambda_p x_p} \sum \lambda_\alpha dh_\alpha \qquad (12\text{-}2)$$

we must have identically

$$\sum \lambda_\alpha \, dh_\alpha = 0 \tag{12-3}$$

hence, with β being fixed

$$\sum_{\alpha=1}^{3} \lambda_\alpha \varphi_{\alpha\beta}(\lambda) = 0$$

When this condition is applied to the components (12-1) of the isotropic spectral tensor, it is seen that

$$Ak^2 + B = 0$$

We put

$$A = -\frac{F}{4\pi k^4} \qquad B = \frac{F}{4\pi k^2}$$

hence

$$\varphi_{\alpha\beta} = \frac{F(k)}{4\pi k^2}\left(\delta_{\alpha\beta} - \frac{\lambda_\alpha \lambda_\beta}{k^2}\right) \tag{12-4}$$

$F(k)$ is the spectral function introduced by Heisenberg. It will be interpreted later.

II. With Kampé de Fériet (reference 26) we first express incompressibility without disturbing the isotropy.

The $\varphi_{\alpha\beta}$ are subject to the condition that the hermitian form

$$\psi = \sum X_\alpha X_\beta^* \, \varphi_{\alpha\beta} \tag{12-5}$$

is positive or zero. It is positive, except when the vector X_α is proportional to vector λ_α, since

$$\sum_{\alpha\beta} \lambda_\alpha \lambda_\beta \varphi_{\alpha\beta} = 0 \tag{12-6}$$

on account of the incompressibility condition (12-3).

The hermitian form ψ is now reduced. According to classical theory and the equation in S, three real numbers S_1, S_2, S_3 and three linear forms

$$Y_1 = \sum a_\alpha X_\alpha \qquad Y_2 = \sum b_\alpha X_\alpha \qquad Y_3 = \sum c_\alpha X_\alpha \tag{12-7}$$

can be found with which ψ can be expressed in the form

$$\psi = S_1 \left|Y_1\right|^2 + S_2 \left|Y_2\right|^2 + S_3 \left|Y_3\right|^2$$

the three vectors of components $a_\alpha, b_\alpha, c_\alpha$ being two by two orthogonal. If $\psi \geq 0$, the numbers S_1, S_2, S_3 are positive or zero.

But, there exists a particular vector γ_α, transformed from vector λ_α by (12-7), so that $\psi = 0$. This can happen only when the three terms of which ψ is the sum, are zero at the same time. Hence two separate hypotheses

$$\text{(a)} \quad S_2 = 0 \qquad S_3 = 0 \qquad \sum a_\alpha \lambda_\alpha = 0$$

$$\psi = S_1 \left|\sum a_\alpha X_\alpha\right|^2 \qquad \varphi_{\alpha\beta} = S_1 a_\alpha^* a_\beta \tag{12-8}$$

a_α being the function of λ. This form of $\varphi_{\alpha\beta}$, similar to (11-8), is too restrictive;

$$(b)\quad S_3 = 0 \qquad \sum a_\alpha \lambda_\alpha = 0 \qquad \sum b_\alpha \lambda_\alpha = 0$$

$$\psi = S_1 \left|\sum a_\alpha X_\alpha\right|^2 + S_2 \left|\sum b_\alpha X_\alpha\right|^2 \tag{12-9}$$

We write rather

$$\psi = S_1 \sum_{\alpha\beta} a_\alpha^* a_\beta X_\alpha^* X_\beta + S_2 \sum_{\alpha\beta} b_\alpha^* b_\beta X_\alpha^* X_\beta$$

and identify with the original form (12-5) of ψ. It is seen that

$$\varphi_{\alpha\beta} = S_1 a_\alpha^* a_\beta + S_2 b_\alpha^* b_\beta \tag{12-10}$$

This is the most general form of a spectral tensor in an incompressible medium. In more simple form

$$\sqrt{S_1}\, a_\alpha = a'_\alpha \qquad \sqrt{S_2}\, b_\alpha = b'_\alpha$$

There are two vectors a'_α, b'_α forming with vector λ_α a trirectangular trihedral and such that

$$\varphi_{\alpha\beta} = a'^*_\alpha a'_\beta + b'^*_\alpha b'_\beta \tag{12-11}$$

By Pythagorean theorem

$$\sum \left|X_\alpha\right|^2 = \frac{1}{k^2}\left|\sum \lambda_\alpha X_\alpha\right|^2 + \frac{1}{a'^2}\left|\sum a'_\alpha X_\alpha\right|^2 + \frac{1}{b'^2}\left|\sum b'_\alpha X_\alpha\right|^2 \tag{12-12}$$

with

$$a'^2 = \sum a'^2_\alpha \qquad b'^2 = \sum b'^2_\beta$$

By this formula, the b'_α can be eliminated from the expression of $\varphi_{\alpha\beta}$

$$\varphi_{\alpha\beta} = \left(1 - \frac{b'^2}{a'^2}\right) a'_\alpha {}^* a'_\beta + b'^2\left(\delta_{\alpha\beta} - \frac{\lambda_\alpha \lambda_\beta}{k^2}\right) \tag{12-13}$$

Finally we put

$$b'^2 = \frac{F(k)}{4\pi k^2} \qquad \sqrt{1 - \frac{b'^2}{a'^2}}\, a_\alpha = c_\alpha$$

This is the canonical form which Kampé de Fériet has given for the spectral tensor of an incompressible fluid

$$\varphi_{\alpha\beta}(\lambda) = c_\alpha {}^*(\lambda) c_\beta(\lambda) + \frac{F(k)}{4\pi k^2}\left(\delta_{\alpha\beta} - \frac{\lambda_\alpha \lambda_\beta}{k^2}\right) \tag{12-14}$$

with

$$\sum c_\alpha(\lambda)\lambda_\alpha = 0$$

The isotropy stipulates that $c_\alpha = 0$, so that formula (12-4) is obtained again.

13. Energy interpretation of the spectral function $F(k)$:

Except when stated otherwise, the isotropic turbulence is assumed to be in the incompressible state.

Hence

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = u_0^2$$

The total energy of the turbulent fluctuations per unit mass is

$$E = \frac{1}{2}\sum \overline{u_\alpha^2} = \frac{1}{2}\sum R_{\alpha\alpha}(0) = \frac{1}{2}R(0) = \frac{3}{2}u_0^2 = \frac{1}{2}\int_\Lambda \sum \varphi_{\alpha\alpha}(\lambda)d\lambda \tag{13-1}$$

But

$$\sum \varphi_{\alpha\alpha}(\lambda) = \frac{F(k)}{2\pi k^2} \tag{13-2}$$

Hence

$$E = \frac{1}{4\pi}\int_\Lambda \frac{F(k)}{k^2}\, d\lambda \tag{13-3}$$

In passing to the polar coordinates in the space of the wave numbers one finds that

$$E = \int_0^\infty F(k)dk \tag{13-4}$$

Which, after minor changes, gives the amount of the energy of the turbulent fluctuations per unit mass in the "sphere of the wave numbers" $\sum \lambda_\alpha^2 < k^2$ as

$$E_k = \int_0^k F(k')dk' \tag{13-5}$$

k' being an integration variable.

The "small values of k" correspond, of course, to the large scales (large vortices).

The next problem is to find the expression of the energy ϵ dissipated as heat per unit mass. While the preceding calculations were by nature strictly kinematic, the dissipative function here must be so introduced that it yields equations of motion, that is, Navier equations; hence, an assumption associated with the dynamics of turbulence, which is discussed in chapter III.

With ν denoting the coefficient of kinematic viscosity, the energy dissipated in heat by the molecular motion is

$$\nu\left[2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)^2 + \ldots\right] \tag{13-6}$$

This is to be expressed in spectral terms and averaged. If

$$u_\alpha = \int_\Lambda e^{i\sum_{p=1}^{3}\lambda_p x_p}\, dh_\alpha(\lambda)$$

we get

$$\frac{\partial u_\alpha}{\partial x_\beta} = i\int_\Lambda \lambda_\beta e^{i\sum_{p=1}^{3}} dh_\alpha(\lambda)$$

and

$$\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} = i \int_\Lambda e^{i \sum_{p=1}^{3} \lambda_p x_p} \left(\lambda_\beta \, dh_\alpha + \lambda_\alpha \, dh_\beta\right)$$

which, squared, reads

$$\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2 = \int_{\Lambda\times\Lambda} e^{i \sum_{p=1}^{3} \left(\lambda'_p - \lambda_p\right) x_p} \left[\lambda_\beta \, dh_\alpha^*(\lambda) + \lambda_\alpha \, dh_\beta^*(\lambda)\right]\left[\lambda'_\beta \, dh_\alpha(\lambda') + \lambda'_\alpha \, dh_\beta(\lambda')\right] \qquad (13\text{-}7)$$

To obtain the averages, involves operation under the $\int$ sign. The result is zero, except when the points λ and λ' coincide in the space Λ of the wave numbers. It leaves

$$\overline{\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2} = \int_\Lambda \left\{\lambda_\beta^2 \left|dh_\alpha\right|^2 + \lambda_\alpha\lambda_\beta \left[dh_\alpha^* \, dh_\beta + dh_\beta^* \, dh_\alpha\right] + \lambda_\alpha^2 \left|dh_\beta\right|^2\right\}$$

$$= \int_\Lambda \left[\lambda_\beta^2 \varphi_{\alpha\alpha} + \lambda_\alpha\lambda_\beta\left(\varphi_{\alpha\beta} + \varphi_{\beta\alpha}\right) + \lambda_\alpha^2 \varphi_{\beta\beta}\right] d\lambda \qquad (13\text{-}8)$$

Consequently, if ϵ is the mean dissipation of energy per unit mass

$$\epsilon = \nu \int_\Lambda \left[k^2 \sum_\alpha \varphi_{\alpha\alpha} + \sum_{\alpha\beta} \lambda_\alpha\lambda_\beta\varphi_{\alpha\beta}\right] d\lambda \qquad (13\text{-}9)$$

This formula supposes neither incompressibility nor isotropy. If, now, the turbulence is incompressible, $\sum_{\alpha\beta} \lambda_\alpha \lambda_\beta \varphi_{\alpha\beta} = 0$. Owing to the isotropy, $\sum \varphi_{\alpha\alpha} = \frac{F}{2\pi k}$.

Hence, by integrating in polar coordinates in space Λ

$$\epsilon = 2\nu \int_0^\infty k^2 F(k)\,dk \tag{13-10}$$

ϵ being a physical quantity, hence finite, it is seen that the preceding integral is finite, which gives a first limitation of the possible forms of the functions $F(k)$.

14. Relations between spectral function $F(k)$ and correlation functions $f(r)$ and $g(r)$:

The spectral tensor $\varphi_{\alpha\beta}$ was defined by the formulas

$$R_{\alpha\beta}(h) = \overline{u_\alpha(x) u_\beta(x + h)} = \int_\Lambda \varphi_{\alpha\beta}(\lambda) e^{i \sum \lambda_p \xi_p} d\lambda$$

which assume homogeneity only. Incompressibility and isotropy are to be added.

According to section 8

$$R_{\alpha\beta}(\xi) = u_0^2 \left[\frac{f - g}{r^2} \xi_\alpha \xi_\beta + g \delta_{\alpha\beta} \right] \qquad g = f + \frac{r}{2} \frac{\partial f}{\partial r}$$

and according to section 12

$$\varphi_{\alpha\beta}(\lambda) = \frac{F(k)}{4\pi k^2} \left(\delta_{\alpha\beta} - \frac{\lambda_\alpha \lambda_\beta}{k^2} \right)$$

By Fourier's reciprocity formulas

$$\varphi_{\alpha\beta}(\lambda) = \frac{1}{(2\pi)^3}\int R_{\alpha\beta}(\xi)e^{-i\sum_{p=1}^{3}\lambda_p\xi_p}\,d\xi \qquad (14\text{-}1)$$

the integral being extended over the entire space. The general formulas for defining F in terms of f or g, and f, g in terms of F must be particularized.

It is pointed out that $\frac{F}{2\pi k^2} = \sum \varphi_{\alpha\alpha}$ is the scalar invariant of the spectral tensor $\varphi_{\alpha\beta}$ ("trace of tensor" or contracted tensor). Hence it seems logical to compare with F the analogous invariant of the correlation vector $R_{\alpha\beta}$, defined by

$$R(r) = \sum R_{\alpha\alpha} = u_0^2(f + 2g) = u_0^2(3f + rf')$$

Thus the following problems must be solved:

(a) express $F(k)$ by means of $R(r)$, and conversely

(b) express $f(r)$ and $g(r)$ by means of $R(r)$

(c) express $f(r)$ and $g(r)$ by means of $F(k)$

Also to be defined in precise manner is what is called the transversal (lateral) and the longitudinal spectrum of turbulence.

(a) Relation between $F(k)$ and $R(r)$:

$$R(r) = \int_\Lambda e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\sum \varphi_{\alpha\alpha}\,d\lambda = \int_\Lambda e^{i\sum_{p=1}^{3}\lambda_p\xi_p}\frac{F(k)}{2\pi k^2}\,d\lambda \qquad (14\text{-}2)$$

To reduce the triple integral to a simple integral, simply select a new axis of the λ_3 perpendicular to the plane $\sum \lambda_p \xi_p = 0$; it results in the integral

$$R(r) = \int_\Lambda e^{ir\lambda_3} \frac{F(k)}{2\pi k^2} d\lambda = \frac{1}{2\pi} \int_\Lambda e^{irk \cos \theta} F(k) \sin \theta \, dk \, d\theta \, d\varphi \tag{14-3}$$

Which, after a short calculation, yields

$$R(r) = 2 \int_0^\infty \frac{\sin rk}{rk} F(k) dk \tag{14-4}$$

Conversely

$$\frac{F(k)}{k} = \frac{1}{\pi} \int_0^\infty r \sin rk R(r) dr \tag{14-5}$$

(b) Relations between $f(r)$, $g(r)$, and $R(r)$:

The integration of the differential equation yields

$$rf' + 3f = \frac{R}{u_0^2} \tag{14-6}$$

or

$$\frac{d}{dr}\left(r^3 f\right) = \frac{r^2 R}{u_0^2}$$

with the condition $f(0) = 0$. It is

$$u_0^2 f(r) = \frac{1}{r^3}\int_0^r r'^2 R(r')dr' \tag{14-7}$$

$R(r)$ having for limit $3u_0^2$, when r tends toward zero, it confirms that the integral is equivalent to $u_0^2 r^3$, and that $f(r)$ tends rather toward unity.

From that, it is deduced, by formula (9-7) that

$$u_0^2 g(r) = \frac{1}{2}R(r) - \frac{1}{2r^3}\int_0^r r'^2 R(r')dr' \tag{14-8}$$

(c) Relations between $f(r)$, $g(r)$, and $F(k)$:

A simple calculation gives

$$\left.\begin{aligned} u_0^2 f(r) &= 2\int_0^\infty F(k)\left(\frac{\sin rk}{r^3k^3} - \frac{\cos rk}{r^2k^2}\right)dk \\ u_0^2 g(r) &= \int_0^\infty F(k)\left(\frac{\cos rk}{r^2k^2} - \frac{\sin rk}{r^3k^3} + \frac{\sin rk}{rk}\right)dk \end{aligned}\right\} \tag{14-9}$$

From that the first terms of the limited developments of $R(r)$, $f(r)$, and $g(r)$ for small values of r are derived in terms of the dissipated energy ϵ

$$\left.\begin{aligned} R(r) &= 3u_0^2\left(1 - \frac{1}{18}r^2\frac{\epsilon}{\nu u_0^2} + \cdots\right) \\ f(r) &= 1 - \frac{1}{30}r^2\frac{\epsilon}{\nu u_0^2} + \cdots \\ g(r) &= 1 - \frac{1}{15}r^2\frac{\epsilon}{\nu u_0^2} + \cdots \end{aligned}\right\} \qquad (14\text{-}10)$$

Comparison with (9-9) indicates that the dissipation length λ is related to the dissipated energy ϵ, the viscosity ν and the kinetic energy of the turbulent fluctuations $E = \frac{3}{2}u_0^2$ through the formula

$$\lambda^2 = 10\nu\frac{E}{\epsilon} = 5\frac{\int_0^\infty F(k)dk}{\int_0^\infty k^2F(k)dk} \qquad (14\text{-}11)$$

15. Lateral and longitudinal spectrum:

The spectral measurements do not provide the spectral function F directly, but merely the spectral terms corresponding to certain simple associations of velocity components, for the observers placed in the particular relative positions. It concerns longitudinal components of the velocity (parallel to the velocity of the main flow), at points placed either on a parallel to the axis or on a perpendicular to the axis. The corresponding correlations are $u_0^2f(r)$ and $u_0^2g(r)$.

According to the original notations of G. I. Taylor

$$\left.\begin{aligned} u_0^2 f(r) &= \int_0^\infty \cos \omega r A(\omega) d\omega \\ u_0^2 g(r) &= \int_0^\infty \cos \omega r B(\omega) d\omega \end{aligned}\right\} \tag{15-1}$$

where $A(\omega)$ and $B(\omega)$ are termed the longitudinal and lateral spectral functions. They now must be expressed by means of $F(k)$.

Since $g = f + \frac{r}{2}f'$, elementary calculation shows that

$$B(\omega) = \frac{1}{2}\left[A(\omega) - \omega A'(\omega)\right] \tag{15-2}$$

and at the same time that

$$R(r) = 2\int_0^\infty \frac{\sin rk}{rk} F(k)dk = \int_0^\infty \cos \omega r \left[A(\omega) + 2B(\omega)\right] d\omega \tag{15-3}$$

Elementary Fourier transformations yield

$$A(\omega) + 2B(\omega) = 2\int_\omega^\infty \frac{F(k)}{k} dk \tag{15-4}$$

A and B are determined by the two equations

$$\left.\begin{aligned} 2B &= A - \omega A' \\ A + 2B &= 2\int_0^\infty \frac{F(k)}{k} dk \end{aligned}\right\} \tag{15-5}$$

The result is

$$A(\omega) = \int_{\omega}^{\infty} \frac{F(k)}{k}\left(1 - \frac{\omega^2}{k^2}\right)dk \qquad B(\omega) = \frac{1}{2}\int_{\omega}^{\infty} \frac{F(k)}{k^2}\left(1 + \frac{\omega^2}{k^2}\right)dk \tag{15-6}$$

CHAPTER III

DYNAMICS OF TURBULENCE

16. Introduction:

To construct a dynamics of turbulence, that is, to set up the differential or finite laws which govern the development of the statistical quantities characterizing the turbulence in time and space, it is necessary to start from elementary laws and apply the statistical methods to them. The most natural idea, the only one which actually produces concrete results, consists in utilizing the Navier equations. To what extent are they applicable to turbulence? The turbulent motion is always a macroscopic motion with respect to a finer scale motion, and, at the limit, with respect to the molecular disturbance. Therefore it is reasonable to believe that it satisfies the equations of the mechanics of fluids. The next step is to find the solution of the Navier equations which, for certain limiting conditions, have the turbulent aspect, and calculate the particular averages from these solutions. Unfortunately, it is rather difficult to define these limiting conditions. So, the remaining resource is to examine whether, among all the possible solutions of the Navier equations, there exist any sufficiently complicated for representing the turbulence, without attempting to determine them logically by the limiting conditions. But then a new difficulty arises. Solutions for the Navier equations are known only for simple conditions which are far from resembling turbulence. In other words, while conceding their validity, we practically do not know how to solve them.

Since it is not acceptable to take the averages on the solutions of the Navier equations, it is attempted to take the averages on the differential equations themselves and to write the differential equations which verify the statistical quantities. Since this method has produced results, it is set forth in the simplest case, that of the Kármán correlation tensor.

17. Fundamental equation of turbulent dynamics:

It is assumed that the total velocity U is uniform and along the axis of x_1. The Navier equations read then

$$\frac{\partial u_\alpha}{\partial t} + U\frac{\partial u_\alpha}{\partial x_1} + \sum_\beta u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu \Delta u_\alpha \qquad (17\text{-}1)$$

in the absence of external forces. ρ is the density, usually assumed constant, and p the pressure. These equations must be supplemented by the equation of continuity

$$\sum \frac{\partial u_\alpha}{\partial x_\alpha} = 0 \qquad (17\text{-}2)$$

The notations are abbreviated by representing the velocity[7] at point $x'_\alpha = x_\alpha + \xi_\alpha$ by $u'_\alpha = u_\alpha(x') = u_\alpha(x + \xi)$. Multiplication of (17-1) by u'_α and summation with respect to α, gives

$$\sum_\alpha u'_\alpha \frac{\partial u_\alpha}{\partial t} + U \sum_\alpha u'_\alpha \frac{\partial u_\alpha}{\partial x_1} + \sum_{\alpha\beta} u'_\alpha u_\beta \frac{\partial u_\alpha}{\partial x_\beta} =$$

$$-\frac{1}{\rho}\sum u'_\alpha \frac{\partial p}{\partial x_\alpha} + \nu \sum_{\alpha\beta} u'_\alpha \frac{\partial^2 u_\alpha}{\partial x_\beta^2} \qquad (17\text{-}3)$$

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[7] In chapter I, section 6, $\overline{u_1}, \overline{u_2}, \overline{u_3}$ represented the velocity components of the whole and u'_1, u'_2, u'_3, those of the velocity of fluctuations. The over-all velocity having then U, o, o, as components, the notations are changed; u_i represents the velocity fluctuations at point x, and u'_i the velocity fluctuations at point x'.

Permutation of the points x and x' yields a similar equation, which is added to the preceding one

$$\frac{\partial}{\partial t}\sum_{\alpha} u_\alpha u'_\alpha + U\sum_{\alpha}\left(u'_\alpha\frac{\partial u_\alpha}{\partial x_1} + u_\alpha\frac{\partial u'_\alpha}{\partial x'_1}\right) + \sum_{\alpha\beta}\left(u'_\alpha u_\beta\frac{\partial u_\alpha}{\partial x_\beta} + u_\alpha u'_\beta\frac{\partial u'_\alpha}{\partial x'_\beta}\right) =$$

$$-\frac{1}{\rho}\sum\left(u'_\alpha\frac{\partial p}{\partial x_\alpha} + u_\alpha\frac{\partial p'}{\partial x'_\alpha}\right) + \nu\sum_{\alpha\beta}\left(u'_\alpha\frac{\partial^2 u_\alpha}{\partial x_\beta^2} + u_\alpha\frac{\partial^2 u'_\alpha}{\partial x'^2_\beta}\right)$$

(17-4)

Taking the averages of the two members it is then assumed that the turbulence is homogeneous, nothing more. The scalar product $\sum \overline{u_\alpha u'_\alpha}$ is a function of ξ_α, and is not individually dependent on x_α and x'_α. Consequently

$$\frac{\partial}{\partial x_1}\sum \overline{u_\alpha u'_\alpha} = -\frac{\partial}{\partial \xi_1}\sum \overline{u_\alpha u'_\alpha} \qquad \frac{\partial}{\partial x'_1}\sum \overline{u_\alpha u'_\alpha} = \frac{\partial}{\partial \xi_1}\sum \overline{u_\alpha u'_\alpha}$$

The term containing the over-all velocity U disappears. The following term can be written

$$\sum_{\alpha\beta} u'_\alpha\frac{\partial}{\partial x_\beta}u_\alpha u_\beta + \sum_{\alpha\beta} u_\alpha\frac{\partial}{\partial x'_\beta}u'_\alpha u'_\beta - \sum_{\beta}\frac{\partial u_\beta}{\partial x_\beta}\sum_{\alpha} u_\alpha u'_\alpha -$$

$$\sum_{\beta}\frac{\partial u'_\beta}{\partial x'_\beta}\sum_{\alpha} u_\alpha u'_\alpha$$

The last two terms disappear as a result of the equation of continuity. $\sum \overline{u_\alpha \frac{\partial}{\partial x_\beta} u_\alpha u_\beta}$ is written:[8]

$$\sum_\beta \frac{\partial}{\partial x_\beta} \sum_\alpha \overline{u'_\alpha u_\alpha u_\beta} = - \sum_\beta \frac{\partial}{\partial \xi_\beta} \sum_\alpha T_{\alpha\beta\alpha}(\xi)$$

Likewise, as a result of the homogeneity, $\sum \overline{u_\alpha \frac{\partial}{\partial x'_\beta} u'_\alpha u'_\beta}$ becomes

$$\sum_\beta \frac{\partial}{\partial x'_\beta} \sum_\alpha \overline{u_\alpha u'_\alpha u'_\beta} = + \sum_\beta \frac{\partial}{\partial \xi_\beta} \sum_\alpha T_{\alpha\beta\alpha}(-\xi)$$

Since $T_{\alpha\beta\gamma}(-\xi) = -T_{\alpha\beta\gamma}(\xi)$, the system of these two terms is equal to

$$-2 \sum_\beta \frac{\partial}{\partial \xi_\beta} \sum_\alpha T_{\alpha\beta\alpha}(\xi) \qquad (17\text{-}5)$$

The viscosity term reads

$$2\nu \sum_\beta \frac{\partial^2}{\partial \xi_\beta^2} \sum_\alpha \overline{u_\alpha u'_\alpha} = 2\nu \, \Delta \sum_\alpha R_{\alpha\alpha} - 2\nu \, \Delta R \qquad (17\text{-}6)$$

where Δ is the Laplacian symbol in three-dimensional space.

Lastly, it is shown that the pressure terms disappear. This is done by extending the definition of the homogeneity to the averages containing the pressure, with due regard to the incompressibility. The averages such as $\overline{pu'_\alpha}$ are not functions separately of x and x' but merely of the difference $\xi = x' - x$, so that, on these functions, $\frac{\partial}{\partial x'} = - \frac{\partial}{\partial x}$.

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[8] See the definition and the properties of the tensors $R_{\alpha\beta}$ and $T_{\alpha\beta\gamma}$ in chapter II, sections 8 and 9.

Therefore, u'_α being function of x'

$$\sum \overline{u'_\alpha \frac{\partial p}{\partial x_\alpha}} = \sum \frac{\partial}{\partial x_\alpha} \overline{p u'_\alpha} = -\sum \frac{\partial}{\partial x'_\alpha} \overline{p u'_\alpha} = -\overline{p \sum \frac{\partial u'_\alpha}{\partial x'_\alpha}} = 0 \tag{17-7}$$

Likewise for the other pressure term. The final result is the fundamental equation (reference 30)

$$\frac{\partial}{\partial t} \sum_\alpha R_{\alpha\alpha} = 2\nu \Delta \sum_\alpha R_{\alpha\alpha} + 2 \sum_\beta \frac{\partial}{\partial \xi_\beta} \sum_\alpha T_{\alpha\beta\alpha} \tag{17-8}$$

It was written with the aid of the components of the tensors $R_{\alpha\beta}$ and $T_{\alpha\beta\gamma}$. Introducing the scalar $R = \sum R_{\alpha\alpha}$ and the vector $\vec{T}$ which has for components

$$T_\beta = \sum_\alpha T_{\alpha\beta\alpha} \tag{17-9}$$

The equation (17-8) assumes the form (reference 2)

$$\frac{\partial R}{\partial t} = 2\nu \Delta R + \operatorname{div} \vec{T} \tag{17-10}$$

18. Case of isotropic turbulence:

The equation (17-10) assumes incompressibility and homogeneity but no isotropy. If the turbulence is isotropic, it is known (14-6) that

$$R(r) = \frac{u_0^2}{r^2} \frac{\partial}{\partial r}(r^3 f)$$

f being the first of the two Kármán correlation functions.

According to the properties of $T_{\alpha\beta\gamma}$ (section 9):

$$T_\beta = 2u_0^3\frac{\xi_\beta}{r}(c + 2a) = u_0^3\xi_\beta\left(c' + \frac{4c}{r}\right) \tag{18-1}$$

The divergence of this vector is equal to

$$\operatorname{div}\vec{T} = \frac{u_0^3}{r^2}\frac{\partial}{\partial r}\left[r^3\left(c' + \frac{4c}{r}\right)\right] \tag{18-2}$$

The fundamental equation (17-10) is now transcribed:

$$\frac{1}{r2}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial t}\left(u_0^2f\right)\right] = \frac{2\nu}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{u_0^3}{r2}\frac{\partial}{\partial r}\left[r^3\left(c' + \frac{4c}{r}\right)\right]$$

In fact, it is remembered that the Laplacian of a function $\varphi(r)$ in n-dimensional space is

$$\Delta_n\varphi(r) = \varphi'' + \frac{n-1}{r}\varphi' = \frac{1}{r^{n-1}}\left(r^{n-1}\varphi'\right)' \tag{18-3}$$

Multiplication by r^2 is followed by integration with respect to r. It is easily verified that all the terms cancel out for $r = 0$. Bearing in mind that

$$r^2\frac{\partial R}{\partial r} = u_0^2r^2\frac{\partial}{\partial r}\frac{1}{r^3}\left(r^3f\right)' = u_0^2r^3\left(f'' + \frac{4}{r}f'\right) \tag{18-4}$$

one arrives at the equation

$$\frac{\partial}{\partial t}\left(u_0^2f\right) = 2\nu u_0^2\left(f'' + \frac{4}{r}f'\right) + u_0^3\left(c' + \frac{4c}{r}\right) \tag{18-5}$$

This is the Kármán-Howarth equation (ref. 30) in its classical form.

The general equation (17-10) has the form of the equation of heat propagation in ordinary three-dimensional space, the quantity being propagated is the scalar R, the invariant of the double correlation tensor. The "second member" $\operatorname{div} \vec{T}$ is tied to the triple correlations as R is to the double correlations. The result is an equation of partial derivatives between two unknown functions R and $\operatorname{div} \vec{T}$. For reasons which appear later on, R is usually considered as the principal unknown. The problem of solving (17-10) consists in defining R by means of suitable physical hypotheses on the triple correlations. The equation with two unknowns finally arrived at is the result of the nonlinear character of the Navier equations. Its mode of operation will be explained later.

The "isotropic" equation (18-5) leaves no trace of the original properties of tensorial symmetry. Its purpose is to connect the functions f and c, which can be measured directly. If it is borne in mind that

$$\left.\begin{aligned} f''(r) + \frac{4}{r} f'(r) &= \Delta_5 f(r) \\ c' + 4\frac{c}{r} &= \frac{1}{r^4} \frac{\partial}{\partial r}\left(r^4 c\right) \end{aligned}\right\} \tag{18-6}$$

then one may write:

$$\frac{\partial}{\partial t}\left(u_0^2 f\right) = 2\nu \Delta_5\left(u_0^2 f\right) + \frac{u_0^3}{r^4} \frac{\partial}{\partial r}\left(r^4 c\right) \tag{18-7}$$

It is an "equation of heat," isotropic naturally, but in a fictitious five-dimensional space. This remark is interesting, but it seems artificial and without real physical significance.

Nevertheless, an unusual property of the functions $u_0(t)$ and $f(r,t)$ can be derived from it. Multiplying by $r^4 dr$ followed by integration from 0 to ∞ yields

$$\frac{\partial}{\partial t} u_0^2 \int_0^\infty r^4 f \, dr = 0 \tag{18-8}$$

provided that $r^4 f'$ and $r^4 c$ tend toward zero when $r \longrightarrow \infty$, which seems reasonable, and difficult to check. Consequently

$$u_0^2(t) \int_0^\infty r^4 f(r,t) dr \tag{18-9}$$

is a numerical constant, independent of t, during the development of the turbulence. It is the Loïtsiansky invariant, which connects the energy of fluctuation $E = \frac{3}{2} u_0^2$ to the correlations f.

It also should be noted that, since the second term of (17-10) is the divergence of 2ν grad $R + \overrightarrow{T}$, its integral, on a sphere of center O and infinitely large radius, is zero. Consequently, the integral of $\frac{\partial R}{\partial t}$ is zero, and

$$\int R \, dx_1 \, dx_2 \, dx_3 = C^{te} \tag{18-10}$$

the triple integral being extended over the entire space. If the turbulence is isotropic, it leaves

$$\int_0^\infty r^2 R(r,t) dr = C^{te} \tag{18-11}$$

But, according to the expression of $R(r)$, remembered at the beginning of this paragraph:

$$\int_0^\infty r^2 R \, dr = u_0^2 \left[r^3 f \right]_0^\infty = u_0^2 \lim_{r \longrightarrow \infty} \left(r^3 f \right)$$

If $r^4f' \longrightarrow 0$, r^3f tends toward zero also, and this limit is zero

$$\int_0^\infty r^2R\ dr = 0 \tag{18-12}$$

This proves that the function R, which is equal to $3u_0^2$ for $r = 0$, and to zero for $r = \infty$, takes negative values for sufficiently high values of r. If formula (18-12) is approximated to formula (14-5) which expresses F(h) by means of R(r), it will be found that, providing R(r) tends sufficiently fast toward zero at infinity, the development of F(k) in powers of k starts with a term in k^4; near $k = 0$ (large size eddies)

$$F(k) = Ck^4 \tag{18-13}$$

with

$$C = -\frac{1}{6\pi}\int_0^\infty r^4R(r)dr$$

The constant C can be expressed by f, because, according to (14-7)

$$u_0^2\int_0^\infty r^4f(r)dr = \int_0^\infty r\ dr\int_0^r r'^2R(r')dr'$$

In order to change the order of integration, the integrals of the second member are written in the form (fig. 5)

$$\lim_{r\longrightarrow\infty}\int_0^r r''\ dr''\int_0^{r''} r'^2R(r')dr' = \lim_{r\longrightarrow\infty}\int_0^r r'^2R(r')dr'\int_{r''}^{r'} r\ dr$$

$$= \frac{1}{2}\lim_{r\longrightarrow\infty} r^2\int_0^r r'^2R(r')dr' -$$

$$\frac{1}{2}\int_0^\infty r'^4R(r')dr'$$

Now:

$$r^2 \int_0^r r'^2 R(r')dr' = u_0^2 r^5 f(r)$$

So, if $f(r)$ tends toward zero at infinity fast enough so that not only $r^3 f$ but even $r^5 f$ approaches zero, then

$$C = \frac{u_0^2}{3\pi} \int_0^\infty r^4 f \, dr \tag{18-14}$$

To a factor $\frac{1}{3\pi}$, C is therefore identical with Loïtsiansky's invariant (18-9).

These results could be extended to include the nonisotropic turbulence.

Lastly it should be noted (ref. 2) that, in the case of isotropic turbulence, the vector $\overrightarrow{T}$ defined by (18-1) is radial, that is to say, that the T_β are proportional to the ξ_β. In effect:

$$T_\beta = \frac{u_0^3(c + 2a)}{r} \xi_\beta \tag{18-15}$$

Hence one may put:

$$\overrightarrow{T} = T \operatorname{grad} r \tag{18-16}$$

$T(r) = u_0^3(c + 2a)$ being a scalar.

19. Local form of the fundamental equation:

The problem is to ascertain what the equation (17-10) becomes when $r \longrightarrow 0$.

It is known that $f(r) \longrightarrow 1$. As $f = 1 - \frac{r^2}{2\lambda^2} + \ldots$, $f'' + \frac{4}{r}f'$ approaches $-\frac{5}{\lambda^2}$.

Lastly, as c is of the third order with respect to r, $c' + \frac{4c}{r}$ tends toward zero.

The final equation, due to G. I. Taylor, reads

$$\frac{d}{dt}u_0^2 = -10\nu\frac{u_0^2}{\lambda^2} \tag{19-1}$$

Recalling that, according to paragraph 14:

$$u_0^2 = \frac{2}{3}E \qquad \lambda^2 = 10\nu\frac{E}{\epsilon}$$

The formula (19-1) becomes therefore[9]

$$\frac{dE}{dt} = -\epsilon \tag{19-2}$$

It establishes an elementary relationship between the dissipation of energy ϵ by viscosity and the decay as a function of time of the energy of fluctuation E; the turbulent energy of fluctuation is totally dissipated as heat by viscosity, this dissipation following a process to be analyzed in chapter V.

The foregoing demonstration stipulates isotropic turbulence. But the result (19-2) is valid in more general cases. In fact, (19-2) can be derived from (17-10). First it is known that $R(0) = 2E$. The connection between ΔR and ϵ is established by the intermediary of the spectral functions $\varphi_{\alpha\beta}(\lambda)$. If the flow is incompressible, formula (13-9) reads

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[9] E being function of the sole variable t, $\frac{\partial}{\partial t}$ can be replaced by $\frac{d}{dt}$, a notation which represents an ordinary derivative with respect to time (not a derivative with respect to motion which would be without sense here).

$$\epsilon = \nu \int_{\Lambda} k^2 \sum \varphi_{\alpha\alpha}(\lambda)d\lambda \tag{19-3}$$

but on the other hand,

$$R(\xi) = \sum R_{\alpha\alpha}(\xi) = \int_{\Lambda} e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \sum \varphi_{\alpha\alpha}(\lambda)d\lambda$$

consequently:

$$\Delta R = - \int_{\Lambda} k^2 e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \sum \varphi_{\alpha\alpha}(\lambda)d\lambda \tag{19-4}$$

and in particular, when ξ_p approaches zero, $\nu\Delta R$ is reduced to

$$-\nu \int_{\Lambda} k^2 \sum \varphi_{\alpha\alpha}(\lambda)d\lambda = -\epsilon.$$

To prove that $\operatorname{div} \overrightarrow{T} = 0$, at the limit, the simplest way is to revert to the Navier equations.

It is plain that equation (19-2), limit of (17-10) when $r \longrightarrow 0$ can be obtained without going by way of the finite values of r. Simply multiply the Navier equations by u_1, u_2, u_3, add them up and average. Assuming the disappearance of the pressure terms to have been proved, it is

$$\frac{1}{2} \frac{d}{dt} \sum \overline{u_\alpha^2} + \sum \overline{u_\alpha u_\beta \frac{\partial u_\alpha}{\partial x_\beta}} = \nu \sum \overline{u_\alpha \frac{\partial^2 u_\alpha}{\partial x_\beta^2}} \tag{19-5}$$

The term $\frac{1}{2}\frac{d}{dt}\sum\overline{u_\alpha^2}$ is written $\frac{dE}{dt}$. The second member, according to the foregoing proof, is equal to $-\epsilon$. Hence it must be shown that $\sum\overline{u_\alpha u_\beta\frac{\partial u_\alpha}{\partial x_\beta}} = 0$. This quantity reads

$$\frac{1}{2}\sum_\beta\overline{u_\beta\frac{\partial}{\partial x_\beta}\sum u_\alpha^2} = \frac{1}{2}\sum_{\alpha\beta}\frac{\partial}{\partial x_\beta}\overline{u_\alpha^2 u_\beta} - \frac{1}{2}\sum_\alpha\overline{u_\alpha^2\sum_\beta\frac{\partial u_\beta}{\partial x_\beta}} \tag{19-6}$$

By virtue of the homogeneity, $\overline{u_\alpha^2 u_\beta}$ is not dependent on x, and the first term is zero.

Owing to the incompressibility, the second term is zero. The first member is therefore zero, which provides the proof.

20. Solution of the fundamental equation, when the triple correlations are disregarded:

When the triple correlations are discounted, equation (17-10) becomes the equation of heat

$$\frac{\partial R}{\partial t} = 2\nu\,\Delta R \tag{20-1}$$

This hypothesis, which mathematically is convenient, is physically quite difficult to justify. Its accuracy increases as the viscous effects become greater (prevalence of term $2\nu\,\Delta R$ over the term $\operatorname{div}\overrightarrow{T}$), or as the Reynolds numbers decrease (whatever their definition). It also would be verified if the velocity components followed the Laplace-Gaussian law. And, what is more interesting, it may be added that it corresponds to a state of turbulence in which the forces of inertia are negligible against the forces of viscosity.

Equation (20-1) must thus be resolved knowing R for $t = 0$, and being aware that, if $r \longrightarrow 0$, R has for limit a finite quantity $3u_0^2(t)$ temporarily considered as known.

In isotropic turbulence, R depends only on r, t, and (20-1) reads

$$\frac{\partial R}{\partial t} = 2\nu\left(\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right) \tag{20-2}$$

To satisfy the limiting conditions, elementary solutions of the form

$$R(r,t) = R_1(t)R_2(s) \qquad s = \frac{r}{\sqrt{8\nu t}} \tag{20-3}$$

are necessary. Hence

$$\frac{tR'_1(t)}{R_1(t)} = \frac{1}{2R_2(s)}\left[\frac{R''_2(s)}{2} + \left(s + \frac{1}{s}\right)R'_2(s)\right] \tag{20-3}$$

The two terms of this equation have a constant value $-\frac{Q}{4}$. Therefore, R_2 satisfies the linear differential equation

$$sR''_2 + 2\left(s^2 + 1\right)R'_2 + QsR_2 = 0 \tag{20-4}$$

or, reduced to classical form, by the transformation,

$$sR_2 = e^{-s^2}H \tag{20-5}$$

Quantity H confirms in fact the Hermitian differential equation

$$H'' - 2sH' + (Q - 4)H = 0 \tag{20-6}$$

Since $R(r,t)$ must cancel out for r infinite, hence also $H(s)$ for s infinite, $Q - 4$ is necessarily an even integer $2n$. In that case H is the nth Hermitian polynomial:

$$H_n(s) = e^{s^2}\frac{d^n}{ds^n}e^{-s^2} \tag{20-7}$$

and

$$R_2(s) = \frac{1}{s}e^{-s^2}H_n(s) \tag{20-8}$$

But R_2 must have a finite limit when $s \longrightarrow 0$, which requires that $H_n(s)$ be an odd polynomial, hence that n be an odd integer: $n = 2p - 1$, $Q = 4p + 2$. The function $R_1(t)$ is then equal to $\frac{1}{t^{p+\frac{1}{2}}}$. Hence the elementary solution:

$$R(r,t) = \frac{1}{t^p}e^{-\frac{r^2}{8\nu t}}\frac{1}{r}H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \tag{20-9}$$

The general solution is a superposition of elementary solutions:

$$R(r,t) = \frac{1}{r}e^{-\frac{r^2}{8\nu t}}\sum_{p=1}^{\infty}\frac{A_p}{t^p}H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \tag{20-10}$$

the constants A_p being so chosen that the series converges and $R(0,t)$ is equal to $3u_0^2$.

Examples: 1. Limited to the term in $p = 2$ it is found that, by interpreting the Hermitian polynomial, $H_3(s)$, and denoting an indeterminate constant by A:

$$R(r,t) = \frac{A}{t^{\frac{5}{2}}}\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}} \qquad (20\text{-}11)$$

hence

$$f(r,t) = e^{-\frac{r^2}{8\nu t}} \qquad u_0^2(t) = At^{-\frac{5}{2}}$$

This example is rational. It is compatible with the hypotheses made at that time. In particular, $r^m f$ approaches zero when $r \longrightarrow \infty$, whatever the non-negative number m may be. However, $f = 0$ for $t = 0$, except when $r = 0$. At the initial instant the turbulence is concentrated at one point from which it ultimately spreads throughout the entire fluid. Such a turbulent structure seems rather difficult to conceive when assuming it to start from $t = 0$. An interpretation will be given later (section 34) in connection with the final phase of decay of turbulence behind a grid.

For $p = 1$, one would have

$$R(r,t) = \frac{A}{t^{\frac{3}{2}}}e^{-\frac{r^2}{8\nu t}} \qquad f = \frac{3}{r^3}\int_0^r r'^2 e^{-\frac{r'^2}{8\nu t}}dr' \qquad (20\text{-}12)$$

f would not converge toward zero when $r \longrightarrow \infty$, which proves that this example cannot be suitable for a real motion.

2. (Kármán-Howarth (ref. 30). Suppose the correlation function f does not depend separately on r and t but only on the variable $s = \frac{r}{\sqrt{8\nu t}}$.

Since

$$R = u_0^2\left(3f + r\frac{\partial f}{\partial r}\right)$$

it then yields

$$R = u_0^2(t)\left[3f(s) + sf'(s)\right] \tag{20-13}$$

so that R is the product of a function of t by a function of s. Thus R is an elementary solution of the problem treated in the first example, corresponding to a chosen Hermitian function H_{2p-1}.

The function $f(s)$ satisfies a differential equation similar to (20-4) written by Kármán and Howarth. But a great advantage accrues from the use of function R and reduction to Hermitian polynomials.

Remark. According to (20-10), $R(r,t)$ is a sum of "elementary solutions" corresponding to various values of the integer p, starting from $p = 2$. For each one of these solutions, except for $p = 2$, it is verified that the invariant of Löitsiansky (18-10) is zero. For $p = 2$ it has a finite value.

21. Solutions involving a similarity hypothesis (see ref. 7):

If the triple correlations are no longer negligible, the fundamental equation cannot be solved completely. It is assumed that the correlation functions $f(r,t)$ and $c(r,t)$ are not separately dependent on the two variables r,t, but solely on $\psi = \frac{r}{l(t)}$ where $l(t)$ is a length in terms of time. That is to say, that at each instant t the correlation curves are superposable, by means of a unit change on the axis of r. However, it is recognized that this hypothesis, called "total similarity" is often too specific, and it is therefore replaced by a "partial similarity" which is verified only in a finite interval $l_1 < r < l_2$.

For total similarity, the Loïtsiansky invariant reads

$$u_0^2(t)\int_0^\infty r^4 f(r,t)\,dr = u_0^2(t)l^5(t)\int_0^\infty \psi^4 f(\psi)\,d\psi \tag{21-1}$$

Consequently, $u_0^2(t)l^5(t)$ is a constant independent of t in the course of the motion. This is no longer true in the case of partial similarity.

To transform the fundamental equation, it is possible to put

$$\left.\begin{aligned} R(r,t) &= 3u_0^2(t)\alpha(\psi) \\ T(r,t) &= 3u_0^3(t)\beta(\psi) \\ \underline{R}_l &= \frac{u_0 l}{\nu} \end{aligned}\right\} \qquad (21\text{-}2)$$

$\alpha(\psi)$ is a function that takes the value unity for $\psi = 0$; $\beta(\psi)$ is a function that cancels out with ψ. $\underline{R}_l$ is a Reynolds number in terms of the time, and T is the scalar which, according to (18-14), defines the triple correlations.

Considering the relation (19-1) which affords $\frac{d}{dt}u_0^2$ in terms of u_0^2 and of the length λ, the fundamental equation

$$\frac{\partial R}{\partial t} = 2\nu \Delta R + \operatorname{div} \vec{T}$$

is easily transformed to

$$\frac{1}{4\nu}\left(l^2\right)'\psi\alpha'(\psi) + 5\frac{l^2}{\lambda^2}\alpha(\psi) + \alpha''(\psi) + \frac{2}{\psi}\alpha'(\psi) + \frac{1}{2}\underline{R}_l\left[\beta'(\psi) + \frac{2}{\psi}\beta(\psi)\right] = 0 \qquad (21\text{-}3)$$

or, schematically arranged:

$$\alpha_1(t)\beta_1(\psi) + \alpha_2(t)\beta_2(\psi) + \alpha_3(t)\beta_3(\psi) + \alpha_4(t)\beta_4(\psi) = 0 \qquad (21\text{-}4)$$

Equation (21-3) is a sum of four terms each of which is the product of a function of t by a function of ψ. How can an equation such as (21-4) be resolved?

The $\alpha_i(t)$ are considered as the coordinates of a point A in four-dimensional space, this point describing a curve (A) in terms of the parameter t. The same applies to the $\beta_i(\psi)$. It results in two curves (fig. 6) such that the straight lines joining the origin to any point A of the first curve and to any point B of the second are always perpendicular. These curves must fit two orthogonal complementary subspaces in four-dimensional space. Hence there are, a priori, three possible cases:

1. (A) is a line passing through the origin. (B) is in the complementary three-dimensional space orthogonal to this line.

2. (A) and (B) are in two completely orthogonal planes passing through the origin.

3. (A) is in a three-dimensional linear space. (B) is the orthogonal line.

In the first case, the $\alpha_i(t)$ are proportional to constant numbers m_i.

In returning to the notations of equation (21-3), the quantities

$$\left(l^2\right)' \qquad \frac{l^2}{\lambda^2} \qquad 1 \qquad \underline{R}_l$$

are proportional to constant numbers. In other words, l^2, $\frac{l^2}{\lambda^2}$, and $\underline{R}_l$ are constant. Following this, the functions $\alpha(\psi)$ and $\beta(\psi)$ are connected by a unique differential equation. Obviously one may assume $l = \lambda$. Then λ^2 is a linear function of time and $u_0\lambda$ is constant. Considering equation (19-1), which we recall here

$$\frac{d}{dt}u_0^2 = -10\nu\frac{u_0^2}{\lambda^2} \tag{19-1}$$

we obtain of necessity:

$$l^2 = \lambda^2 = 10\nu t \qquad u_0^2 = \frac{C^{te}}{t} \qquad \underline{R}_\lambda = C^{te} \tag{21-5}$$

The Loïtsiansky relation is not verified, so that no total similarity can prevail; $\alpha(\psi)$ and $\beta(\psi)$ are joined by the relation

$$\alpha'' + \left(\frac{5}{2}\psi + \frac{2}{\psi}\right)\alpha' + 5\alpha + \frac{1}{2}R_\lambda\left(\beta' + \frac{2}{\psi}\beta\right) = 0 \qquad (21\text{-}6)$$

which can be subjected to experimental check.

A complete discussion of the second and third possibility is foregone, in favor of the case where $l = \lambda$ is prescribed. Then (21-3) becomes

$$\frac{1}{4\nu}\left(\lambda^2\right)'\psi\alpha'(\psi) + \alpha''(\psi) + \frac{2}{\psi}\alpha'(\psi) + 5\alpha(\psi) + \frac{1}{2}R_\lambda\left[\beta'(\psi) + \frac{2}{\psi}\beta(\psi)\right] = 0 \qquad (21\text{-}7)$$

and is represented schematically by

$$\alpha_1(t)\beta_1(\psi) + \alpha_2(t)\beta_2(\psi) + \alpha_3(t)\beta_3(\psi) = 0 \qquad (21\text{-}7)$$

The discussion then deals with three-dimensional rather than four-dimensional space.

Two cases are possible:

(1) The point $\alpha_i(t)$ describes a straight line (issuing from 0), and point $\beta_i(\psi)$ remains in the plane perpendicular to this line (passing through 0).

(2) The roles of the two points are permutated.

The first case is not distinct from that which we have studied. In the second case, there exist three constants m_1, m_2, m_3, so that

$$\left.\begin{aligned} &\frac{\psi\alpha'(\psi)}{4\nu m_1} = \frac{\alpha''(\psi) + \frac{2}{\psi}\alpha'(\psi) + 5\alpha(\psi)}{m_2} = \frac{\beta'(\psi) + \frac{2}{\psi}\beta(\psi)}{2m_3} \\ &m_1\left(\lambda^2\right)' + m_2 + m_3\underline{R}_\lambda = 0 \end{aligned}\right\} \qquad (21\text{-}8)$$

They are the conditions proposed by Sedov. They can be checked directly by experiment; m_1 cannot be zero, because $\underline{R}_\lambda$ would be = Cte, and at the same time $\alpha(\psi) = C^{te}$, which is not possible; m_2 and m_3 cannot be zero simultaneously.

If $m_3 \neq 0$, the similarity cannot be total, because one would have $u_0{}^2\lambda^5 = C^{te}$ and by (19-1)

$$\lambda^2 = 4\nu t \qquad u_0{}^2 t^{5/2} = C^{te} \qquad \underline{R}_\lambda t^{3/4} = C^{te}$$

which is incompatible with the second equation (21-8).

Following this, u_0 and λ are solutions of the system of differential equations:

$$\left.\begin{aligned} &m_1\left(\lambda^2\right)' + m_2 + \frac{m_3}{\nu}u_0\lambda = 0 \\ &\left(u_0{}^2\right)' = -10\nu\frac{u_0{}^2}{\lambda^2} \end{aligned}\right\} \qquad (21\text{-}9)$$

$\alpha(\psi)$ verifies the second-order differential equation:

$$\alpha'' + \left(\frac{2}{\psi} - \frac{m_2}{4\nu m_1}\psi\right)\alpha' + 5\alpha = 0 \qquad (21\text{-}10)$$

after which β is determined by the first-order differential equation

$$\beta' + \frac{2}{\psi}\beta = \frac{m_3}{2\nu m_1}\psi\alpha' \tag{21-11}$$

The system (21-9) can be reduced to quadratures. If the solution cannot be written in finite terms, it still is desirable to integrate as far as possible.

When the first equation (21-9) is set in the form

$$\frac{\underline{R}}{\underline{R}_0} = \frac{1 - k}{10\nu}\left(\lambda^2\right)' + k \tag{21-12}$$

where $\underline{R}_0$ and k are constants replacing m_1, m_2, and m_3, it is found that

$$\lambda = C^{te}\underline{R}^{1-k}\left(\underline{R} - \underline{R}_0\right)^k \tag{21-13}$$

Whence it is deduced that

$$\frac{d\underline{R}}{dt} = C^{te}\frac{\underline{R} - k\underline{R}_0}{\underline{R} - (1 - k)\underline{R}_0}\left(1 - \frac{\underline{R}_0}{\underline{R}}\right)^{1-2k} \tag{21-14}$$

The problem is thus reduced to a quadrature. By (21-13), λ follows from $\underline{R}$, and $u_0 = \nu\frac{\underline{R}}{\lambda}$.

The equation (21-10) generalizes equation (20-4) to a certain extent, but it is less simple and will not be discussed.

Other types of solutions satisfying a similarity hypothesis can be examined, such as the case where λ is very small, that is, where the dissipation due to viscosity is low. In that case a length other than λ must be chosen for length l, such as $L = \int_0^\infty f(r)dr$. The discussion

proceeds on the assumption that the viscosity terms of the fundamental equation are negligible against the other two. It leaves then

$$\frac{dR}{dt} = \operatorname{div} \overrightarrow{T} \tag{21-15}$$

The calculations are not developed (for further details, consult Batchelor's report (ref. 7)). The sole purpose in this paragraph was to give an idea of the various methods that can be applied to solve the fundamental equation on the basis of the similarity hypothesis. The physical study of the problem is deferred until later.

22. Transformation of the fundamental equation in spectral terms:

The formulas of reciprocity between the correlations and the spectral functions made it seem interesting to transcribe the fundamental equation (17-10) in spectral terms. (Compare spectral intermedium, section 19.)

By (11-10):

$$R = \sum R_{\alpha\alpha} = \int_{\Lambda} e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \left(\sum \varphi_{\alpha\alpha} \right) d\lambda = \int e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \varphi(\lambda) d\lambda \tag{22-1}$$

with

$$\varphi\left(\lambda_1, \lambda_2, \lambda_3\right) = \sum \varphi_{\alpha\alpha}$$

Likewise

$$\Delta R = -\int e^{i \sum_{p=1}^{3} \lambda_p \xi_p} k^2 \varphi d\lambda \tag{22-2}$$

Hence

$$\frac{\partial R}{\partial t} - 2\nu \Delta R = \int e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \left(\frac{\partial \varphi}{\partial t} + 2\nu k^2 \varphi \right) d\lambda \qquad (22\text{-}3)$$

This quantity is equal to $\operatorname{div} \overrightarrow{T}$, where $\overrightarrow{T}$ is the vector having for components $\sum_{\alpha} T_{\alpha\beta\alpha}$. We put:

$$\operatorname{div} \overrightarrow{T} = \int_{\Lambda} \psi(\lambda) e^{i \sum_{p=1}^{3} \lambda_p \xi_p} d\lambda$$

The fundamental equation becomes

$$\frac{\partial \varphi}{\partial t} + 2\nu k^2 \varphi = \psi \qquad (22\text{-}4)$$

In this equation φ and ψ are unknown functions of λ, t, and $k^2 = \sum_{p=1}^{3} \lambda_p^2$.

The total energy of the fluctuation is

$$E = \frac{1}{2} \int_{\Lambda} \varphi d\lambda$$

(it was shown in section 9 that $\int_{\Lambda} \psi d\lambda = 0$).

For isotropic turbulences, the spectral function $F(k) = 2\pi k^2 \varphi$, which does not depend separately on $\lambda_1, \lambda_2, \lambda_3$ as in the general case for φ, but solely on k, is introduced.

The factor ψ can also be expressed by means of the scalar

$$T = u_0^3(c + 2a) = \frac{u_0^3}{2} r\left(c' + \frac{4}{r}c\right) = \frac{u_0^3}{2r^3}\left(cr^4\right)'$$

It is actually known that

$$\operatorname{div} \overrightarrow{T} = T' + \frac{2}{r}T = \frac{1}{r^2}\left(r^2T\right)' \tag{22-5}$$

which, after elementary calculations, gives

$$\psi(k) = \frac{1}{2\pi^2}\int_0^\infty \frac{\sin rk}{rk}\left(r^2T\right)' \, dr \tag{22-6}$$

Equation (22-4) reads

$$\frac{\partial F}{\partial t} + 2\nu k^2 F = 2\pi k^2 \psi \tag{22-7}$$

Putting: $2\pi k^2 \psi = \Psi(k,t)$, where Ψ is a function connected with triple correlations and such that

$$\int_0^\infty \Psi \, dk = 0$$

yields

$$\frac{\partial F}{\partial t} + 2\nu k^2 F = \Psi \tag{22-8}$$

By the use of the formula (22-5) and the limited development of sin rk, it is readily proved that the function $\Psi(k)$ is infinitely small as k^6, provided that the correlation function c approaches

zero faster at infinity than $\frac{1}{r^4}$. But, with the relation $\int_0^\infty \Psi \, dk = 0$, nothing more is known a priori about its form.

Equation (22-8) is not essentially different from (17-10). It raises the same difficulties and is always a single equation between two unknowns F and Ψ. However, it is much easier to make mathematically or physically reasonable assumptions about the form of Ψ than about the form of the triple correlations, as will be shown in the following two methods, both due to Heisenberg, which take advantage of (22-8); the first involves the introduction of the correlations in time and will go beyond the purpose originally assigned to it, the second leads effectively to certain possible forms of the spectral function F.

23. First theory of Heisenberg (ref. 23):

The hypotheses to be examined primarily are those which form the subject of the last part of Heisenberg's report. We shall utilize the ideas and follow the calculations as closely as possible, but change the notations and substitute stochastic averages[10] for the spatial and temporal averages.

The expressions giving the statistical velocity and pressure of homogeneous turbulence read:

$$u_\alpha = \int_\Lambda e^{i \sum_{\beta=1}^{3} \lambda_\beta x_\beta} \, dh_\alpha(\lambda) \qquad p = \int_\Lambda e^{i \sum_{\beta=1}^{3} \lambda_\beta x_\beta} \, dq(\lambda) \qquad (23\text{-}1)$$

The equations of motion are first written in spectral terms. It can be assumed that $h_\alpha(\lambda)$ is derivable on each test with respect to the time, and its derivative, which corresponds to $\frac{\partial u_\alpha}{\partial t}$, is designated by $h_\alpha(\lambda)$.

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[10] Bass, reference 5. The calculations are rather difficult. The reader who wants to avoid reading them will find a summary of the results at the end of paragraph 24.

The expressions (23-1) must be introduced in the equations (23-2)

$$\left.\begin{aligned} &\frac{\partial u_\alpha}{\partial t} + \sum_\beta u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + U\frac{\partial u_\alpha}{\partial x_1} = -\frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu \sum_\beta \frac{\partial^2 u_\alpha}{\partial x_\beta^2} \\ &\sum_\alpha \frac{\partial u_\alpha}{\partial x_\alpha} = 0 \end{aligned}\right\} \qquad (23\text{-}2)$$

where U is the constant over-all velocity.

The only term of which the equation merits a more detailed discussion is

$$u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -i \int_{\Lambda\times\Lambda} \lambda'_\beta e^{-i\sum \lambda'_p x_p} e^{i\sum \mu_p x_p} dh_\alpha^*(\lambda') dh_\beta(\mu) \qquad (23\text{-}3)$$

Putting $\mu_p - \lambda'_p = \lambda_p$ and considering the equation of continuity yields

$$\sum \lambda_\alpha \, dh_\alpha(\lambda) = 0$$

which had already been used in the form

$$u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = i\lambda_\beta \int_{(\mu)} e^{i\sum \lambda_p x_p} dh_\alpha^*(\mu - \lambda) dh_\beta(\mu) \qquad (23\text{-}4)$$

the subscript (μ) specifying that the integral is extended over the values of μ. This quantity is the Fourier transform with respect to λ of

$$i\lambda_\beta \int_{(\mu)} dh_\alpha^*(\mu - \lambda) dh_\beta(\mu)$$

By means of some elementary transformations two equations are obtained:

$$\left.\begin{aligned} &\dot{dh}_\alpha(\lambda) + i\sum_\beta \lambda_\beta \int_{(\mu)} dh_\alpha^*(\mu - \lambda)dh_\beta(\mu) + i\lambda_1 U\, dh_\alpha(\lambda) = \\ &-\frac{i}{\rho}\lambda_\alpha\, dq(\lambda) - \nu k^2 dh_\alpha(\lambda) \\ &\frac{dq(\lambda)}{\rho} = -\frac{1}{k^2}\sum_{\alpha\beta} \lambda_\alpha\lambda_\beta \int_\mu dh_\alpha^*(\mu - \lambda)dh_\beta(\mu) \end{aligned}\right\} \quad (23\text{-}5)$$

where $\dot{h}_\alpha(\lambda)$ designates the derivative of $h_\alpha(\lambda)$ with respect to λ.

The demonstration of paragraph 17 from equations (23-5) is then resumed. The first equation (23-5) is multiplied by $dh_\alpha^*(\lambda)$, added up with respect to α, i is then changed to $-i$ and the two equations obtained are added member by member.

Putting

$$Z = -2i\sum_{\alpha\beta} \lambda_\beta\, dh_\alpha^*(\lambda) \int_{(\mu)} dh_\alpha^*(\mu - \lambda)dh_\beta(\mu) \quad (23\text{-}6)$$

while considering the equation of continuity, gives

$$\frac{\partial}{\partial t}\sum \left|dh_\alpha(\lambda)\right|^2 + 2\nu k^2 \sum \left|dh_\alpha(\lambda)\right|^2 = \underline{R}(Z) \quad (23\text{-}7)$$

$\underline{R}(Z)$ denotes the real part of Z.

After averaging, it is known that when the turbulence is isotropic:

$$\sum \overline{\left|dh_\alpha(\lambda)\right|^2} = \frac{F}{2\pi k^2}\, d\lambda$$

Consequently

$$\frac{d\lambda}{2\pi k^2}\left(\frac{\partial F}{\partial t} + 2\nu k^2 F\right) = \underline{\overline{R(Z)}} \tag{23-8}$$

What was called Ψ in paragraph 22 is given by the equation

$$2\pi k^2 \underline{\overline{R(Z)}} = \Psi \, d\lambda \tag{23-9}$$

In order to find a useful expression of Z several hypotheses originating with Heisenberg are made.

First of all, it is known that the statistical functions $dh_\alpha(\lambda)$ are of orthogonal increments, that is, that the averages of the products $dh_\alpha(\lambda)dh_\alpha^*(\lambda')$ are zero, if the two points λ,λ' of the wave number space are distinct. The first hypothesis is that the statistical variables $dh_\alpha(\lambda)$, $dh_\alpha^*(\lambda')$ are not only orthogonal but independent (in probability).

Recalling that the average values $\overline{dh_\alpha(\lambda)}$ are zero and that $dh_\alpha^*(\lambda) = dh_\alpha(-\lambda)$, it is seen that the average value of

$$dh_\alpha^*(\lambda)dh_\alpha^*(\mu - \lambda)dh_\beta(\mu)$$

is zero, except when at the same time

$$\lambda = \lambda - \mu \qquad \mu - \lambda = \mu \qquad \lambda = \mu$$

that is, $\lambda = \mu = 0$. If it is assumed that the probabilities of sudden jumps of $h_\alpha(\lambda)$ are zero, then $Z = 0$.

Paragraph 20 dealt with the case where $Z = 0$. It is certain that, in general, Z is not rigorously zero, but it is likely that Z is small compared to the viscosity term, or the term of development $\frac{\partial F}{\partial t}$. In other words, the hypothesis of orthogonality is too general, and that of the

independence too restrictive, yet it is difficult to formulate an intermediary hypothesis. But, starting from the independence it is possible to obtain, by means of certain interesting transformations, a significant expression for Z. This expression is defective since there is contradiction. But it seems likely that it represents a good approximation of Z. We shall therefore study it.

24. First theory of Heisenberg (continued). Space-time correlations:

If the average of $\underline{R}(Z)$ disappears, it is because Z contains an odd number of factors $dh_\alpha(\lambda)$; the equations of motion are not linear. Replace $dh_\beta(\mu)$ by the time integral of its derivative $d\dot{h}_\beta(\mu)$ and express $d\dot{h}_\beta(\mu)$ by means of the equations of motion. Certain terms linear in dh are replaced by quadratic terms; products of four factors dh appear in Z, and $\overline{Z}$ is not zero. A kind of technique of solving the equations of motion is involved here, which reduces them to integral equations (bearing on simple averages).

If T is a very great positive number

$$dh_\beta(\mu,t) - dh_\beta(\mu,-T) = \int_{-T}^{t} d\dot{h}_\beta(\mu,\tau)d\tau = \int_{0}^{t+T} d\dot{h}_\beta(\mu,t-\tau)d\tau \tag{24-1}$$

If T is sufficiently large it may be conceded that $h_\beta(\mu,-T)$ tends toward zero, that is, that the fluid has "started from rest." Hence

$$dh_\beta(\mu,t) = \int_{0}^{\infty} d\dot{h}_\beta(\mu,t-\tau)d\tau \tag{24-2}$$

To lighten the notations, $h'_\beta(\mu)$ is to represent the function $h_\beta(\mu,t-\tau)$. Then

$$Z = -2i \sum_{\alpha\beta} \lambda_\beta \, dh_\alpha^*(\lambda) \int_0^\infty d\tau \int_{(\mu)} dh_\alpha^*(\mu-\lambda) d\dot{h}'_\beta(\mu) \tag{24-3}$$

and $d\dot{h}'_\beta(\mu)$ is eliminated by means of the equations of motion from which we removed previously the term of the pressure in dq:

$$d\dot{h}'_\beta(\mu) = -i \sum_\gamma \mu_\gamma \int_{(\mu')} dh'_\beta{}^*(\mu' - \mu) dh'_\gamma(\mu') +$$

$$i\frac{\mu_\beta}{\mu^2} \sum \mu_{\alpha'}\mu_{\beta'} \int_{(\mu')} dh'_{\alpha'}{}^*(\mu' - \mu) dh'_{\beta'}(\mu') - (\nu\mu^2 + i\mu_1 U) dh'_\beta(\mu) \tag{24-4}$$

with $\mu^2 = \sum \mu_\alpha{}^2$.

Now a new assumption is made according to which the original assumption of independence (for $\tau = 0$), which is certainly verified for the great values of τ, remains valid for the whole time interval τ, leaving only the averages with four factors

$$\bar{Z} = -2 \sum_{\alpha\beta\gamma} \lambda_\beta \int_0^\infty d\tau \int_{(\mu,\mu')} \mu_\gamma \overline{dh_\alpha{}^*(\lambda) dh_\alpha{}^*(\mu - \lambda) dh'_\beta{}^*(\mu' - \mu) dh'_\gamma(\mu')} +$$

$$2 \sum_{\alpha\beta\alpha'\beta'} \frac{\lambda_\beta \mu_{\alpha'} \mu_\beta \mu_{\beta'}}{\mu^2} \int_0^\infty d\tau \int_{(\mu,\mu')} \overline{dh_\alpha{}^*(\lambda) dh_\alpha{}^*(\mu - \lambda) dh'_{\alpha'}{}^*(\mu' - \mu) dh'_{\beta'}(\mu')} \tag{24-5}$$

The dh being independent in distinct points of the frequency space, it is necessary to associate the four points which correspond to the four dh by pairs, not forgetting that $dh_\alpha{}^*(\lambda) = dh_\alpha(-\lambda)$. Considering the transformation of the first integral, the only admissible combinations are

$(1)^{st}$

$$\lambda = \lambda - \mu \qquad \mu' - \mu = \mu'$$

hence $\mu = 0$. No contribution to the integral.

$(2)^{nd}$

$$\lambda = \mu - \mu' \qquad \mu - \lambda = \mu'$$

conditions which are reduced to $\mu' = \mu - \lambda$, and leave an integral other than zero

$$\int_{(\mu)} \mu_\gamma \overline{dh_\alpha^*(\lambda)dh'_\beta(\lambda)}\ \overline{dh_\alpha^*(\mu - \lambda)dh'_\gamma(\mu - \lambda)} \tag{24-6}$$

(3)rd

$$\lambda = \mu' \qquad \mu - \lambda = \mu - \mu'$$

conditions which are reduced to $\mu' = \lambda$, and leave behind the integral

$$\int_{(\mu)} \mu_\gamma \overline{dh_\alpha^*(\lambda)dh'_\gamma(\lambda)}\ \overline{dh_\alpha^*(\mu - \lambda)dh'_\beta(\mu - \lambda)} \tag{24-7}$$

After some changes in the denomination of the subscripts, the first term of $\overline{Z}$ assumes the form

$$-2\sum_{\alpha\beta\gamma} \int_0^\infty d\tau \int_{(\mu)} \left(\lambda_\beta\mu_\gamma + \lambda_\gamma\mu_\beta\right) \overline{dh_\alpha^*(\lambda)dh'_\beta(\lambda)}\ \overline{dh_\alpha^*(\mu - \lambda)dh'_\gamma(\mu - \lambda)} \tag{24-8}$$

Putting $\mu - \lambda = \lambda'$, while considering the incompressibility condition, equation (24-8) becomes

$$-2\sum_{\alpha\beta\gamma} \lambda_\gamma\lambda'_\beta \int_0^\infty d\tau \int_{(\lambda')} \overline{dh_\alpha^*(\lambda)dh'_\beta(\lambda)}\ \overline{dh_\alpha^*(\lambda')dh'_\gamma(\lambda')} \tag{24-9}$$

A similar calculation which need not be given in detail gives the expression of the second term of $\overline{Z}$. Finally

$$\overline{Z} = 2\int_0^\infty d\tau \int_{(\lambda')} \frac{k^2 - k'^2}{k^2 + k'^2 + 2\sum_\delta \lambda_\delta\lambda'_\delta}$$

$$\sum_{\alpha\beta\gamma} \lambda_\gamma\lambda'_\beta \overline{dh_\alpha^*(\lambda)dh'_\beta(\lambda)}\ \overline{dh_\alpha^*(\lambda')dh'_\gamma(\lambda')} \tag{24-10}$$

Now the components of the correlation tensor in space and time are introduced:

$$S_{\alpha\beta}(\xi,t,\tau) = \overline{u_\alpha(x,t)u_\beta(x+\xi,t-\tau)} \tag{24-11}$$

The u_α are expressed in spectral terms. Putting

$$\overline{dh_\alpha^*(\lambda)dh'_\beta(\lambda)} = \overline{dh_\alpha^*(\lambda,t)dh_\beta(\lambda,t-\tau)} = \theta_{\alpha\beta}(\lambda,t,\tau)d\lambda \tag{24-12}$$

it is seen that

$$S_{\alpha\beta}(\xi,t,\tau) = \int_\Lambda e^{i\sum \lambda_p \xi_p}\, \theta_{\alpha\beta}(\lambda,t,\tau)d\lambda \tag{24-13}$$

a formula which ties the correlations $S_{\alpha\beta}$ to the generalized spectral tensor $\theta_{\alpha\beta}$. Now, $\bar{Z}$ is expressed by means of $\theta_{\alpha\beta}$:

$$\bar{Z} = 2\,d\lambda \int_0^\infty d\tau \int_{(\lambda')} \frac{k^2 - k'^2}{k^2 + k'^2 + 2\sum \lambda_\delta \lambda'_\delta}$$

$$\sum_{\alpha\beta\gamma} \lambda_\gamma \lambda'_\beta \theta_{\alpha\beta}(\lambda,t,\tau)\theta_{\alpha\gamma}(\lambda',t,\tau)d\lambda' \tag{24-14}$$

For easier reading of this formula, the term under the sign $\int$ represents the scalar product of the two vectors having for components

$$\sum_\beta \lambda'_\beta \theta_{\alpha\beta}(\lambda,t,\tau) \quad \text{and} \quad \sum \lambda_\gamma \theta_{\alpha\gamma}(\lambda',t,\tau)$$

The real part of $\bar{Z}$ is obtained by transforming the $\theta_{\alpha\beta}$ into correlations $S_{\alpha\beta}$ by (24-13) and taking the real parts of the exponentials.

All these calculations do not necessarily assume isotropic turbulence. But in this particular case they are a little simplified and will be terminated. If there is isotropy and incompressibility:

$$\theta_{\alpha\beta}(\lambda,t,\tau) = A(k,t,\tau)\left(\delta_{\alpha\beta} - \frac{\lambda_\alpha\lambda_\beta}{k^2}\right) \tag{24-15}$$

$A = \frac{1}{2}\sum \theta_{\alpha\alpha}$ being a real quantity to be interpreted right away.

A being real, $\overline{Z}$ is real and equal to $\overline{R(Z)}$. Therefore

$$\overline{R(Z)} = 2\,d\lambda \int_0^\infty d\tau \int_{(\lambda')} \frac{k^2 - k'^2}{k^2 + k'^2 + 2\sum \lambda_\delta\lambda'_\delta}$$

$$\sum_{\alpha\beta\gamma} \lambda_\gamma\lambda'_\beta A(k,t,\tau)A(k',t,\tau)\left(\frac{\lambda_\alpha\lambda_\beta}{k^2} - \delta_{\alpha\beta}\right)\left(\frac{\lambda'_\alpha\lambda'_\gamma}{k^2} - \delta_{\alpha\beta}\right) d\lambda' \tag{24-16}$$

To terminate the calculation, the last factors are developed, and with the vector λ taken as support of the third axis one passes to polar coordinates. The angle θ of vectors λ and λ' is introduced, defined by

$$\sum \lambda_\delta\lambda'_\delta = kk' \cos\theta$$

Finally, one puts $\cos\theta = \zeta$ and introduces the elementary function

$$J(k,k') = -4\pi\left(k^2 - k'^2\right)k^3k'^3 \int_{-1}^{1} \frac{\zeta(1 - \zeta^2)}{k^2 + k'^2 + 2kk'\zeta}\,d\zeta$$

$$= \pi\left(k^2 - k'^2\right)\left(k^4 + k'^4 - \frac{2}{3}k^2k'^2\right) -$$

$$\frac{\pi}{2}\frac{\left(k^2 + k'^2\right)\left(k^2 - k'^2\right)^3}{kk'} \log\left|\frac{k + k'}{k - k'}\right| \tag{24-17}$$

Summary and conclusion.- By the successive assumptions of the present paragraph, the fundamental equation (22-8) reads

$$\frac{\partial F}{\partial t} + 2\nu k^2 F = \int_0^\infty d\tau \int_0^\infty J(k,k')A(k,t,\tau)A(k',t,\tau)dk' \qquad (24\text{-}18)$$

The function $J(k,k')$ given by (24-17) changes sign when k and k' are permuted. The integral of the second member of (24-18) with respect to k, from 0 to ∞, is therefore zero and (24-18) contains, as it should be, the known equation (19-2):

$$\frac{\partial E}{\partial t} + \epsilon = 0$$

as it follows from formulas (13-4) and (13-10).

The function $A(k,t,\tau)$ is tied to the space-time correlations:

$$S_{\alpha\beta}(\xi,t,\tau) = \overline{u_\alpha(x,t)u_\beta(x + \xi, t - \tau)}$$

by the relation

$$\sum S_{\alpha\alpha}(\xi,t,\tau) = 2\int e^{i \sum \lambda_p \xi_p} A(k,t,\tau)d\lambda$$

If the turbulence is isotropic it gives, more simply:

$$\sum S_{\alpha\alpha}(\xi,t,\tau) = 8\pi \int_0^\infty k^2 \frac{\sin rk}{rk} A(k,t,\tau)dk \qquad (24\text{-}19)$$

and conversely

$$A(k,t,\tau) = \frac{1}{4\pi^2} \int_0^\infty r^2 \frac{\sin rk}{rk} \left(\sum S_{\alpha\alpha}\right) dr \qquad (24\text{-}20)$$

$\sum S_{\alpha\alpha}$ being naturally a function of r alone.

If $\tau = 0$, the general tensor $S_{\alpha\beta}(\xi,t,\tau)$ is reduced to the correlation tensor in space $R_{\alpha\beta}(\xi,t)$ and $\sum S_{\alpha\alpha}$ becomes $\sum R_{\alpha\alpha} = R$.

$$F(k,t) = 4\pi k^2 A(k,t,0) \tag{24-21}$$

is verified whereby the formulas (24-19) and (24-20) contain the formulas (14-4) and (14-5). Therefore, (24-18) is a nonlinear integral equation, in which appear at the same time $A(k,t,\tau)$ and $A(k,t,0) = F(k,t)$ and which makes it possible in principle to determine the general correlations, in time and space, of isotropic turbulence by means of their spectral function $A(k,t,\tau)$.

25. Second theory of Heisenberg:

This "second theory" came first in Heisenberg's report (ref. 23). By means of postulates of physical character, it effectively enables the form of the spectral function $F(k)$ to be determined in certain conditions. But the formula set up still contains an indeterminate constant. It seems that Heisenberg's idea was to carry the expression obtained for F in the nonlinear equations deduced from the equations of motion, which would have made it possible to calculate the constant. The theory of sections 23 and 24 does not exactly come up to these expectations. It results in a nonlinear integral equation in which not the spectral function $F(k,t)$ but its space-time extension $A(k,t,\tau)$ is involved. If $A(k,t,0) = F(k,t)$ are known, by the theory which is to be discussed, it seems theoretically possible to deduce from it $A(k,t,\tau)$ for every time interval τ. But the mathematical problem thus posed appears difficult and does not fit into any class of known problems, and no attempt is made to solve it.

The dissipation of energy per unit mass in the ensemble of the spectrum is, as is known

$$\epsilon = 2\nu \int_0^\infty k^2 F(k)\,dk \tag{25-1}$$

The dissipated energy per unit mass in the spectral range $\sum \lambda_\alpha^2 < k^2$, that is, by the eddies of "dimensions greater than $\frac{1}{k}$" and of relatively large scale are designated by ϵ_k. If $k \longrightarrow \infty$, ϵ_k tends toward ϵ, and if $k \to 0$, ϵ_k tends toward zero. Heisenberg reduced ϵ_k to a form similar to ϵ by introducing an apparent coefficient of viscosity ν_k so that

$$\epsilon_k = 2(\nu + \nu_k) \int_0^k k'^2 F(k')dk' \tag{25-2}$$

ν_k must tend toward zero when $k \longrightarrow \infty$.

Now this first hypothesis must be perfected in such a way that ν_k assumes a usable form.

Since $\lim_{k \longrightarrow \infty} \nu_k = 0$, ν_k can be represented by an integral $\int_k^\infty$. What about the function under the sign $\int_k^\infty$? It is assumed that this function depends on the values k' ranging between k and ∞ directly and by means of $F(k')$, but that it does not depend on k. In that case, it can be determined by dimensional consideration. Its product by dk' having the dimensions of a viscosity ν, it is necessarily of the form $C\sqrt{\frac{F(k')}{k'^3}}$, where C is a purely numerical constant.

Hence

$$\epsilon_k = 2\left(\nu + C \int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\, dk'\right) \int_0^k k'^2 F(k')dk' \tag{25-3}$$

ϵ_k is the sum of two terms. The first corresponds to the part of the energy of the fluctuations of the spectral range (0,k) which is transformed directly into heat. The second is the part of that energy which, before being transformed into heat, serves, first, to maintain

the energy of the fluctuations of the spectral range (k,∞) of the "small eddies."

Now, the fundamental equation (22-8) can be decomposed precisely in this fashion. Integration member by member from 0 to k yields

$$\frac{\partial}{\partial t}\int_0^k F(k')dk' + 2\nu\int_0^k k'^2F(k')dk' = \int_0^k \Psi(k')dk' \qquad (25\text{-}4)$$

The first member manifests, first, the energy loss of the fluctuations of the range $(0,k)$, then the portion of this energy dissipated directly by viscosity. The second member is therefore the portion which is transformed into energy of the fluctuations of the range (k,∞), and the formula (25-3) gives an expression of the function Ψ. Ultimately

$$\frac{\partial}{\partial t}\int_0^k F(k')dk' + 2\left(\nu + C\int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\,dk'\right)\int_0^k k'^2F(k')dk' = 0 \qquad (25\text{-}5)$$

This is Heisenberg's fundamental equation[11]. For the present, it is not discussed in its most general form, but only in a special case.

Heisenberg's hypotheses are incompatible with the concept of absolute steadiness, because (25-5) has no solution independent of t. In chapter V, it will be shown that Heisenberg's equation can be applied to the problem of spontaneous decay of turbulence behind a grid, an essentially unsteady problem. But, an assumption of partial steadiness can be made as will be shown in chapter IV. For the values of k greater than a fixed approximate value k_0, it can be assumed that the energy dissipated by the large eddies (wave numbers lower than k) is a constant. Only the manner of distribution between the turbulent fluctuations of the small eddies (wave numbers higher than k) and the

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[11]It appears in this form in the second of Heisenberg's reports (reference 24), but the particular case in question is treated in his first report (reference 23).

dissipation by viscosity (transformation into heat, molecular fluctuations) depend on k, the proportion dissipated as heat increasing with k according to formula (25-3).

It will be shown (in chapter IV) that this assumption of partial steadiness is equivalent to an assumption of statistical equilibrium or similarity, which is the more accurate the higher the Reynolds number of the turbulence. On these premises the hypothesis is discussed from the mathematical point of view. Equation (25-5) must be replaced by

$$2\left(\nu + C\int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\,dk'\right)\int_0^k k'^2F(k')dk' = C^{te} = \epsilon_0 \quad \text{if} \quad k > k_0 \tag{25-6}$$

To solve it, simply put

$$\theta(k) = \int_0^k k'^2F(k')dk' \qquad F(k) = \frac{1}{k^2}\theta'(k) \tag{25-7}$$

The function $\theta(k)$ cancels out for $k = 0$ and verifies the equation

$$\nu + C\int_k^\infty \sqrt{\frac{\theta'(k')}{k'^5}}\,dk' = \frac{\epsilon_0}{2\theta(k)} \tag{25-8}$$

which by differentiation becomes the elementary differential equation

$$\frac{\theta'}{\theta^4} = \left(\frac{2C}{\epsilon_0}\right)^2\frac{1}{k^5} \tag{25-9}$$

Putting $H^3 = \frac{4}{3}\frac{\epsilon_0}{2C}$ and introducing an integration constant k_1 it is seen that

$$\theta(k) = Hk^{\frac{4}{3}}\left[1 + \left(\frac{k}{k_1}\right)^4\right]^{-\frac{4}{3}} \tag{25-10}$$

and finally

$$F(k) = F(k_0)\left(\frac{k}{k_0}\right)^{-\frac{5}{3}}\left[\frac{1 + \left(\frac{k}{k_1}\right)^4}{1 + \left(\frac{k_0}{k_1}\right)^4}\right]^{-\frac{4}{3}} \qquad (25\text{-}11)$$

This formula defines the spectrum of isotropic turbulence for $k > k_0$, in terms of an indeterminate parameter k_1, and by means of Heisenberg's assumptions, themselves suggested by Weizsäcker's hypothesis of similarity (reference 42), which will be examined later.

Discussion.- If k is small against k_1, $F(k)$ is reasonably proportional to $k^{-\frac{5}{3}}$. If k is large, $F(k)$ is proportional to k^{-7} (figure 7). These two laws have been indicated by Heisenberg. The first is of special interest and used for deriving the corresponding form of the correlation functions.

The formula (25-11) has been established only when $k > k_0$ and, actually, has no physical significance if $k \longrightarrow 0$, because $F(k)$ cannot become infinite. It is supposed here, in first approximation, that, if $k < k_0$, the true form of $F(k)$ is $F(k) = 0$, and, moreover, that k decidedly is smaller than k_1. To interpret this hypothesis mathematically is the same as replacing k_1 by ∞ in (25-11); hence, one uses the physical limiting conditions in which k_1 is infinite (very high Reynolds numbers).

Putting

$$\left.\begin{aligned} F(k) &= 0 && \text{if } k \leq k_0 \\ &= F(k_0)\left(\frac{k}{k_0}\right)^{-\frac{5}{3}} && \text{if } k \geq k_0 \end{aligned}\right\} \qquad (25\text{-}12)$$

the correlation function $R(r)$ can be computed.

According to (14-4):

$$R(r) = 2F(k_0)k_0^{\frac{5}{3}} \int_{k_0}^{\infty} \frac{\sin rk}{rk} k^{-\frac{5}{3}} dk \qquad (25\text{-}13)$$

This integral is convergent when $k_0 > 0$; it is not derivable twice under the integration sign. An attempt is made to find the form of function $R(r)$ for small values of r. It is shown that there exists a number α such that

$$R(r) = 3u_0^2 \left(1 - Ar^{\alpha}\right)$$

A being a constant.

When $r \longrightarrow 0$, it is easily verified that $R(r)$ has a finite limit, equal to

$$3u_0^2 = 2F(k_0)k_0^{\frac{5}{3}} \int_{k_0}^{\infty} k^{-\frac{5}{3}} dk = 3F(k_0)k_0 \qquad (25\text{-}14)$$

Forming

$$3u_0^2 - R(r) = 2F(k_0)k_0^{\frac{5}{3}} \int_{k_0}^{\infty} \left(1 - \frac{\sin rk}{rk}\right) k^{-\frac{5}{3}} dk \qquad (25\text{-}15)$$

and making the change of the variable $rk = x$ in the integral, leaves

$$r^{\frac{2}{3}} \int_{rk_0}^{\infty} \left(1 - \frac{\sin x}{x}\right) x^{-\frac{5}{3}} dx \qquad (25\text{-}16)$$

Posting

$$I = \int_0^\infty \left(1 - \frac{\sin x}{x}\right) x^{-\frac{5}{3}} dx$$

The integral $\int_{rk_0}^\infty$ differs from I by infinitesimally small quantities with r. Hence, in first approximation when r is not too great,

$$R(r) = 3u_0^2 \left(1 - Ar^{\frac{2}{3}}\right) \qquad (25\text{-}17)$$

with

$$A = \frac{2}{3} k_0^{\frac{2}{3}} I = 0.801\, k_0^{\frac{2}{3}}$$

The exponent looked for, α, is equal to $\frac{2}{3}$. This formula is important, and we shall give later on demonstrations for it, of wholly different appearances. The formula that gives the correlation function $f(r)$ is similar to (25-17). It again is a "law in $r^{\frac{2}{3}}$."

It is a limiting law for infinite k_1. If k_1 is finite, formula (25-11) yields for $R(r)$ a law of the ordinary type

$$R(r) = 3u_0^2 \left(1 - \frac{r^2}{2\lambda^2} + \ldots\right)$$

which proceeds, for $r = 0$, with a horizontal tangent and a finite radius of curvature equal to λ. When $k_1 \longrightarrow \infty$, this radius of curvature tends toward zero and the limiting curve has the form (25-17) (figure 8).

Note.- Comparison of Heisenberg's equations (25-5) and (24-18). Equation (25-5) is an equation in F, which, theoretically, identifies F and can be used to reduce it to a (complicated) differential equation. The calculation was made in the "steady" state. It leaves, in the expression of F, the constant C, which equation (25-5) can obviously not determine. If (24-18) had been an equation in F too, it could have been used to determine C. But the procedure that results in this equation and whose original purpose is to eliminate the triple correlations, meets the expectations only partially, because it replaces in some way the triple correlations by the correlations of space and of time. Equation (24-18) serves therefore, as far as the mathematical problem is accessible, to compute the form of the space-time relations, once the space correlations are determined by (25-5). But it does not appear that the numerical solution of (24-18) has ever been attempted up to now.

CHAPTER IV

THEORY OF LOCAL ISOTROPY AND STATISTICAL EQUILIBRIUM

26. Introduction:

The importance attached to the study of isotropic turbulence was, up to the last few years, justified by considerations of, seemingly, essentially practical value.

The development of turbulence toward isotropy had in its favor a rather feeble theoretical argument, namely, that, far from the sources of turbulence, the Reynolds stresses could be assumed zero (see section 6), which is a condition necessary, but far from being sufficient, for isotropy.

Batchelor (reference 12) proved in 1948 that this argument is of no value; the elements of the anisotropy of turbulence during its formation could be recovered in the final phase of its evolution.

But, from the experimental point of view, the isotropy of turbulence behind a grid seems well established, by numerous experimental works, as will be shown later. But none of these evolved a general concept; it seemed that a particular case was studied here solely by reason of its simplicity.

The theories to be examined here form, by way of contrast, an unusual ensemble, not only because of the successful experimental verifications but particularly because of the scope in which they contribute to further studies of turbulence. They bring to light the characters of turbulent motion which are of general significance.

The most outstanding result is perhaps the better understanding of the role played by the viscosity in turbulence. The viscosity is involved at the birth of turbulence and at its death; at its birth it is the condition necessary for the creation of the rotation and the diffusion of large turbulent masses in the fluid; it intervenes at its death (decay) since the turbulent energy is ultimately transformed into heat; but the complexity of the turbulent motion stems precisely from the fact that the viscosity plays a negligible part during the major part of the transformations of turbulent energy.

The fundamental outline is as follows: relatively large turbulent masses with relatively low velocity gradients are formed in the creative regions of turbulence. The sense of the word relative can be defined by introducing the Reynolds number $\frac{u_0 L_0}{\nu}$, where $\rho\frac{u_0^2}{2}$ is the mean kinetic energy and L_0 the mean quantity of these turbulent masses.[12] In order to have turbulence in the proper sense of the word, the ratio

$$\frac{\rho u_0^2}{\rho\nu\frac{u_0}{L_0}} = \frac{u_0 L_0}{\nu}$$

must be high. If it is small, we find ourselves at once in what Batchelor has called the final period of turbulence, in which the rotation decays, so to speak, on the spot. The forces of inertia play a secondary part. The flow is quasi-laminar. This is the phase through which all turbulence passes. It has been experimentally and theoretically studied by Batchelor and Townsend (references 10, 12), and we shall come back to it later on.

If the Reynolds number is high, the vortices undergo a double evolution, according to Heisenberg's representation (reference 23); they get bigger on the one hand and smaller on the other, that is, their spatial dimensions increase on the one hand, and break up into smaller vortices on the other. Leaving aside the first aspect of this evolution, for the time being, the theory of similarity studies the development of the energy passed on to the small vortices; the latter undergo a development similar to that of the generating vortices, they pass on their energy to smaller vortices and so forth up to the moment where the size of the vortices is such that their Reynolds number is small. A bigger and bigger part of the energy is then changed into heat. This somewhat vague representation is defined a little more accurately further on, in particular, by the analysis of the rotational development.

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[12]This rather vague expression is specified to some extent by the study of dynamic equilibrium (chapter V).

It is reasonable to think - and this assumption will be backed up by its consequences - that the development of vortices is the more rapid the smaller their size. On the other hand, the statistical character of the velocity of the fluctuations, which is prescribed by experience, is assumed.

Hence it may be imagined that, when the energy has been transmitted to vortices sufficiently smaller than the initial vortices, the structure of these vortices is, on the average, independent of that of the generating vortices, and that the relatively slow development of the latter has no effect on the development of the smaller size vortices. The latter, therefore, find themselves in a quasi-balanced state, one determining factor of which is the energy per unit mass and time which is supplied by the bigger to the smaller vortices.

27. Definition of local homogeneity and local isotropy:

At first we shall follow Kolmogoroff's method (reference 31. For more details see Batchelor, reference 6) which seems the most direct as well as the easiest, at least, to begin with. Kolmogoroff introduces quantities which can be completely defined in a limited spatial domain G and a limited temporal field T. Assume that M and M' are two points of G, that t and t' are two instants of t; we call r the distance MM', and put $\tau = t - t'$.

The difference w of the velocities at M and M' at instants t and t' is introduced

$$w_i = u_i(M,t) - u_i(M',t') \qquad (27\text{-}1)$$

and defined by its law of probability in the space-time domain (G,T). One notices immediately that w is little dependent on the spectral components of the velocity which develop slowly in the spatial domain G and in the time interval T.

By its definition w depends a priori on M and t, as well as on M' and t'.

Kolmogoroff defines what he calls the local homogeneity and local isotropy. A condensed version of these definitions reads as follows:

The turbulence is said to be locally homogeneous when the laws of probability for the statistical function w depend, in the domain (G,T),

only on τ and the vector $\overrightarrow{MM'}$ (actually, instead of this vector, the vector $\vec{e}$ defined later on is involved).

The turbulence is termed locally isotropic when it is locally homogeneous and when these probability laws depend neither on the orientation nor on the sense of the coordinate axes.

According to the foregoing, it may be assumed that in every turbulence having a rather high Reynolds number, there exists a spatial domain G which is small compared to the size of the big vortices, and a time interval T, short compared to the time necessary for their energy to change appreciably, and such that the turbulence is locally homogeneous and isotropic in the domain (G,T).

But these definitions can have no physical significance unless the reference system used by the observer is itself physically coordinated with the turbulence. The domain G must, in some fashion, follow the turbulence with the mean flow velocity. Kolmogoroff, in his original report (ref. 31) defines the velocity w about a reference point (M_0,t_0) in terms of the components ξ_i of the vector $\overrightarrow{M_0M}$ and of $\tau = t - t_0$ by the formulas

$$w_i = u_i(M,t) - u_i\left(M_0,t_0\right) \tag{27-2}$$

then he introduces a vector e having as component

$$e_i = \xi_i - \tau u_i\left(M_0,t_0\right) \tag{27-3}$$

This is a random vector whose physical significance is evident in figure 9. Since the essential point of the vector $\tau u\left(M_0,t_0\right)$ arises from the "mean velocity" in the domain G, the use of the coordinates e_i is tantamount to following the turbulence in its entire motion.

The correlation measurements in time relative to the laws of probability of w are few and far between. Some measurements have been made in water and in air, but none in the wind tunnel, as far as we know. Obviously, in such measurements, time-space intervals must be combined and it must be ascertained whether correlations close to 1 can be obtained for a finite value of the ratio $\frac{\overrightarrow{MM'}}{\tau}$ which, if it exists, define. the mean velocity at the scale involved.

But, at the present state of research the study is limited to simultaneous quantities for defining the statistical values w. As in ordinary, isotropic turbulence, it is usually limited to the moments of the second and third order. The tensor of the moments of the second order

$$\overline{w_i(M,M_0)w_j(M',M_0)}$$

simultaneously introduces three points, but can be expressed, on account of the isotropy, homogeneity and incompressibility, as a sum of moments relative to the couples (M,M_0), (M',M_0), (M,M'):

$$\overline{w_i(M,M_0)w_j(M',M_0)} = \frac{1}{2}\left[\overline{w_i(M,M_0)w_j(M,M_0)} + \overline{w_i(M',M_0)w_j(M',M_0)} - \overline{w_i(M,M')w_j(M,M')}\right] \qquad (27\text{-}4)$$

In this case, the tensor

$$B_{ij} = \overline{w_i(M,M')w_j(M,M)'} = \overline{(u'_i - u_i)(u'_j - u_j)}$$

plays a role parallel to that of the tensor R_{ij} in isotropic turbulence, and by the same argument it is apparent that the isotropy leads to writing

$$B_{ij} = \frac{\xi_i\xi_j}{r^2}\left[B_{dd}(r) - B_{nn}(r)\right] + \delta_{ij}B_{nn}(r) \qquad (27\text{-}5)$$

where

$$B_{dd}(r) = \overline{(u'_d - u_d)^2} \qquad B_{nn}(r) = \overline{(u'_n - u_n)^2}$$

u_d and u_n representing the velocity components along MM' or normal to MM'.

A total moment can be defined too:

$$B(r) = \sum_i B_{ii} = \overline{\sum \left(u'_i - u_i\right)^2}$$

The incompressibility involves the relation

$$B_{nn} = B_{dd} + \frac{r}{2} \frac{\partial B_{dd}}{\partial r} \tag{27-6}$$

equivalent to Kármán's relation in isotropic turbulence (9-7). Other relations can be established, in particular, the expression of the dissipation of energy by viscosity

$$\epsilon = \frac{15}{2}\nu \left(\frac{\partial^2 B_{dd}(r)}{\partial r^2} \right)_{r=0} \tag{27-7}$$

Likewise, the moments of the third order can be expressed by the single function

$$B_{ddd}(r) = \overline{\left(u'_d - u_d\right)^3}$$

28. Similarity hypotheses. Statistical equilibrium:

The definition of these local quantities enables the notion of statistical equilibrium of small vortices to be defined. According to our hypotheses, this equilibrium is not dependent on the particular characteristics of the flow. The only quantities which can be of influence are:

(1) The energy supplied per unit mass and time to the small vortices by the larger ones;

(2) The factor governing the kinematic viscosity.

If the fluid is assumed incompressible, the specific mass can be discounted.

Hence Kolmogoroff's postulate of similarity: in a domain where the turbulence is locally homogeneous and isotropic, the laws of probability depend only on r, ϵ, and ν.

This postulate justifies the dimensional analysis which results in setting

$$B_{dd}(r) = \sqrt{\epsilon\nu}\beta_{dd}\left(\frac{r}{l}\right) \tag{28-1}$$

In this formula, β_{dd} is a universal function, and $l = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$ represents a length, called the local scale of turbulence by Kolmogoroff. This length is much smaller than the length of dissipation λ of the isotropic turbulence. It is, in effect (compare (14-11))

$$\epsilon = 15\frac{u_0^2}{\lambda^2}$$

whence

$$\frac{\lambda}{l} = \sqrt{R_\lambda}.\sqrt[4]{15}$$

where the Reynolds number $R_\lambda = \frac{u_0\lambda}{\nu}$ must be at least some tens if the conditions of similarity are to be realized.

The nondimensional statistical quantities must be universal functions of $\frac{r}{l}$. Consequently

$$\left.\begin{aligned}
&\text{(coefficient of dissymmetry)} \qquad S_d\left(\frac{r}{l}\right) = \frac{\overline{\left(u'_d - u_d\right)^3}}{\left[\overline{\left(u'_d - u_d\right)^2}\right]^{\frac{3}{2}}} \\
&\text{(coefficient of flatness)} \qquad A_d\left(\frac{r}{l}\right) = \frac{\overline{\left(u'_d - u_d\right)^4}}{\left[\overline{\left(u'_d - u_d\right)^2}\right]^2}
\end{aligned}\right\} \qquad (28\text{-}2)$$

and similar quantities S_n, A_n, with the normal velocity component u_n.

They are universal functions in a certain domain $l \ll r \ll L$, in which L is a length characterizing the size of G. It is to be noted that, according to the hypotheses, l and L can be slowly variable functions of time. The precise definition of the range of validity of the formulas is ticklish, but in any case, if r is made $= 0$, the values of

$$S_d(0) = \frac{\overline{\left(\frac{\partial u_d}{\partial x}\right)^3}}{\left[\overline{\left(\frac{\partial u_d}{\partial x}\right)^2}\right]^{\frac{3}{2}}} \qquad \text{and} \qquad A_d(0) = \frac{\overline{\left(\frac{\partial u_d}{\partial x}\right)^4}}{\left[\overline{\left(\frac{\partial u_d}{\partial x}\right)^2}\right]^2} \qquad (28\text{-}3)$$

must be universal constants, which affords a first confirmation of the similarity hypothesis.

Townsend (ref. 37, 38), at Cambridge, developed a method of measuring $S(0)$ and $A(0)$. The measurements were first made in isotropic turbulence produced by a grid. These coefficients are constant over a wide range of Reynolds numbers (ratios varying from 1 to 5) and at varying distances from the grid (likewise in a ratio near to 5), that is, at different moments of the development of the turbulence.

Townsend quotes the value 3.49 ± 0.04 for A and -0.38 for S, with a little lower accuracy. It should be recalled that for Gaussian distributions the values of A and S are, respectively, 3 and 0.

To check the theory of locally isotropic turbulence, by which the structure of the small vortices is independent of the nature of the large vortices and the formation of turbulence, Townsend made also measurements in the wake of a cylinder (refs. 39, 40, and 41), that is, in turbulence neither homogeneous nor isotropic (in the original sense). He determined thus $S_d(0)$, $S_n(0)$, $A_d(0)$ and $A_n(0)$ over the entire extent of the wake, at distances of more than 80 diameters from the cylinder. The values obtained are the same, with sufficient accuracy, as those obtained for isotropic turbulence behind a grid. Furthermore, the measurement of the quantities

$$\overline{\left(\frac{\partial u_1}{\partial x}\right)^2} \qquad \overline{\left(\frac{\partial u_2}{\partial x}\right)^2} \qquad \overline{\left(\frac{\partial u_3}{\partial x}\right)^2}$$

u_1 being parallel to the overall velocity (axis of x_1), shows that, as for isotropic turbulence and throughout the entire wake, these quantities satisfy the equations of isotropy

$$\overline{\left(\frac{\partial u_1}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u_2}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u_3}{\partial x}\right)^2}$$

By the same argument, the expression of the energy dissipation is $\frac{15}{2}\nu\overline{\left(\frac{\partial u_1}{\partial x}\right)^2}$, as for isotropic turbulence, and as the equations (27-5) and (27-7) may lead to presume.

These formulas are very important because they show the possibility of defining the dissipation of energy in any turbulent flow by measurements bearing on a single velocity component.

29. Case of high Reynolds numbers (reference 6):

The study of the statistical equilibrium of vortices can be extended further when the Reynolds number is high enough so that the viscous dissipation in the largest vortices of the statistical equilibrium is relatively low; it can be assumed then that the statistical equilibrium of these large vortices is not dependent on ν.

This is Kolmogoroff's second hypothesis:

If $l \ll r \ll L$, the laws of probability regarding w are then dependent on ϵ and r only.

This implies that a domain G can be found whose size L is large compared to l, that is, that the Reynolds number is very high.

Dimensional analysis permits then the definition of the form of the moments of the statistical variable w. In fact it is readily apparent that, for $B_{dd}(r)$ not to be dependent on ν, $\beta\left(\frac{r}{l}\right)$ must be of the form

$$\beta\left(\frac{r}{l}\right) = C^{te} \times \left(\frac{r}{l}\right)^{\frac{2}{3}}$$

whence

$$B_{dd}(r) = C(\epsilon r)^{\frac{2}{3}} \qquad (29\text{-}1)$$

C being a universal constant (figure 10).

The moment B_{nn} must have a similar form. The equation of continuity indicates then that $B_{nn}(r) = \frac{4}{3}B_{dd}(r)$.

On the other hand, when $\frac{r}{l}$ is small,

$$B_{nn}(r) = \frac{1}{2}B''_{nn}(0)r^2 = 2B_{dd}(r) \qquad (29\text{-}2)$$

Likewise, one must have

$$B_{ddd}(r) = C^{te} \times \epsilon r$$

But, if r is small

$$B_{ddd}(r) = C^{te} \times r^3 \tag{29-3}$$

because in that case, $B_{ddd}(r) \neq S_d(0)\left[B_{dd}(r)\right]^{\frac{3}{2}}$, in first approximation. Since l depends on ν, the nondimensional quantities such as $S_d\left(\frac{r}{l}\right)$ must be constant, for values of r much greater than l. On the other hand, $S_d(0)$ is also a universal constant, and the experiment seems to indicate that these two constants are identical.

To tell the truth, there is only one series of measurements of $S_d(r)$ made by Townsend in a region where the turbulence was isotropic, by measuring the triple correlation function

$$c(r) = \frac{\overline{u^2u'}}{u_0^3}$$

The relations

$$\overline{\left(u - u'\right)^3} = -6u_0^3c(r) \qquad \overline{\left(u - u'\right)^2} = u_0^2\left[1 - f(r)\right] \tag{29-4}$$

are utilized.

The experimental points are placed near $S_d = -0.38$ for values of $\frac{r}{l}$ of the order of $40l$. The absolute value of S decreases slowly afterward.

Instead of arguing about the correlation functions, one may just as well apply the dimensional analysis to the spectral functions $F(k)$ for values of k which are sufficiently large without being too large, however. Since $F(k)$ must be dependent only on k and ϵ, it is found that $F(k)$ must be proportional to $\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$ in the domain under consideration.

This is the spectral law in $k^{-\frac{5}{3}}$ spoken of earlier during the discussion of Heisenberg's theory (section 25).

Using the detailed hypotheses of Heisenberg, we have then shown that there is equivalence between the law of correlation in $r^{\frac{2}{3}}$ and the spectral law in $k^{-\frac{5}{3}}$. It is now seen that this equivalence was inevitable, and that it is not the result of the particular form of the law of energy transfer in the spectrum, but solely due to the existence of a general similarity hypothesis among the hypotheses the utilization of which was unimportant.

By using the equations of motion the value of $B_{ddd}(r)$ can be defined and a relation established between the constants C and S, which is susceptible to experimental verifications.

The equation corresponding to that of Kármán (18-5) reads here

$$-4\epsilon = B'_{ddd} + \frac{4}{r}B_{ddd} - 6\nu\left(B''_{dd} + \frac{4}{r}B'_{dd}\right) \qquad (29\text{-}5)$$

The time factor does not appear, because the hypothesis of local homogeneity contains that of steadiness with time, so that $\frac{d}{dt}u_0^2$ must be taken equal to $-\frac{2}{3}\epsilon$.

The equation (29-5) multiplied by r^4, is integrated in the form

$$6\nu B'_{dd} - B_{ddd} = \frac{4}{5}\epsilon r \qquad (29\text{-}6)$$

If r is small, B_{ddd} can be neglected, so that one finds again the relation

$$\epsilon = \frac{15}{2}\nu B''_{dd}(0)$$

For the values of r in the interval $l \ll r \ll L$, B_{ddd} prevails over B'_{dd}, because B_{ddd} is proportional to r and B'_{dd} to $r^{-\frac{1}{3}}$. Hence

$$B_{ddd} = -\frac{4}{5}\epsilon r$$

But in (29-3) it was seen that in first approximation:

$$B_{ddd}(r) = S_d(0)\left[B_{dd}(r)\right]^{\frac{3}{2}} = S_d(0)C^{\frac{3}{2}}\epsilon r$$

Consequently

$$C = \left(-\frac{4}{5S_d(0)}\right)^{\frac{3}{2}} \tag{29-7}$$

Therefore, C can be calculated, if the value of $S_d(0)$ is known and, particularly, if the experimental value $S_d(0) = -0.38$ is assumed. In that case

$$B_{dd}(r) = C(\epsilon r)^{\frac{2}{3}} = 1.64(\epsilon r)^{\frac{2}{3}}$$

Conversely, the experimental study of the function $B_{dd}(r)$ enables C to be measured and the obtained value to be compared with the value computed from $S_d(0)$.

30. Validity of the similarity laws:

All second-order moment measurements have been made in isotropic turbulence. Hence it is useful to examine what becomes of the theory of local isotropy if applied to isotropic turbulence proper.

The first point is the determination of the space-time range of validity. The discussion, necessarily rather vague, has been made by Batchelor (ref. 6). It is briefly summarized here:

From the spatial point of view, isotropy appears to be assured for dimensions of the order of magnitude of the correlation length $L = \int_0^\infty f \, dr$.

The difficulty arises from the estimation of the time interval. An evaluation of the characteristic period of the vortices indicates that the ratio between the period of the largest and that of the smallest vortices is of the order of $\frac{L}{\lambda}$, where λ is the length of dissipation. This ratio must, therefore, be great. Experience indicates an order of magnitude of $\frac{L}{\lambda}$ in what Batchelor calls the "initial phase of decay of turbulence." (It is to be defined in chapter V.) For the present, it is simply stated that it is produced by the turbulence behind a grid of mesh size M, when the Reynolds number $R_M = \frac{UM}{\nu}$ of the grid (U = mean velocity) is sufficiently high (1,000 to 300,000) and the measurements are made at a distance from the grid of the order of 1,000 M. One finds then that $\frac{L}{\lambda}$ is of the order of $\frac{R_\lambda}{10}$, where $R_\lambda = \frac{u_0 \lambda}{\nu}$ may be called the Reynolds number of turbulence. During the phase in question, R_λ is constant and equal to $\sqrt{\frac{R_M}{A}}$, where A is a number dependent on the constitution of the grid.

For values of R_λ not quite up to 200, which still gives only values of ratio $\frac{L}{\lambda}$ of the order of 20, Townsend has measured the correlation function g(r). Since

$$1 - g(r) = \frac{B_{nn}(r)}{2u_0^2}$$

$\frac{1 - g(r)}{r^{\frac{2}{3}}}$ must be constant.

The curves plotted for various values of R_M and $\frac{r}{M}$ have similar shapes. They present a rather flat maximum between $r = 0.1L$ and $r = 0.3L$, from which the value of the constant C (section 29) can be deduced. The average of the obtained values is 1.53, with discrepancies of less than 3 percent. Considering the accuracy of the measurements of $S_d(0)$, the agreement with the value 1.64 deduced from $S_d(0)$ is sufficient.

Other measurements, made at a lower Reynolds number, yield $C = 1.33$, while those by Dryden produced $C = 1.50$. In conclusion, although the range of a validity of the law is too restricted for the aspect of the curves to be convincing, the agreement of the values of C is quite satisfactory.

The lower limit of validity of the law of local isotropy in these experiments is of the order of magnitude of λ, that is, well superior to l, since $\frac{\lambda}{l} = \sqrt[4]{15R_\lambda}$ is here of the order of 20 to 25.

It seems that the Reynolds numbers used are still too low for the field of application of Kolmogoroff's second hypothesis to be extended very far. The first hypothesis, less restrictive, is much better verified. Townsend's correlation measurements show, in fact, that the curves representing $g(r)$ are when $r \ll \lambda$ functions of $\frac{r}{\lambda}$ only, at different distances from the grid. The similarity is extended to values of $\frac{r}{\lambda}$ higher than unity for the highest values of R_λ, where the second hypothesis possesses a domain of validity. This validity is tied to the fact that R_λ is constant at the beginning of the turbulent development. (Compare chapter V.) This information on the development of turbulence can, obviously, not be deduced from the theory of local isotropy which assumes ϵ constant.

On the other hand, the absence of similarity in the correlation curves at high values of $\frac{r}{\lambda}$ suggests that the range of application of the similarity law is very restricted, and also belief in the fact that the ratio of correlation length L to dissipation length λ varies in reasonable fashion during the evolution of turbulence, even when R_λ remains reasonably constant, because the relation $\frac{L}{\lambda} = \frac{R_\lambda}{10}$ is only a first approximation.

In reality, the domain of statistical equilibrium is extremely extended, but this does not appear in the structure of the spatial correlation tensor. For a clear definition of the individual behavior of each

scale of vortex dimensions the introduction of the spectral tensor is necessary. Heisenberg's theory is therefore found capable of exceeding the stage reached by the theory of local isotropy, and it likewise permits interpretation of the different stages of development of turbulence discussed in paragraph 31, and in chapter V.

31. Interpretation of the laws of statistical equilibrium in spectral terms - Weizsäcker's and Heisenberg's theories (refs. 42 and 23, respectively):

Kolmogoroff's theory assumes a locally steady equilibrium and its spatial domain of application is necessarily restricted. However, examination of the correlation curves and of the laws of variation of the correlation length L during the decay of isotropic turbulence behind a grid leads one to believe that the scope of similarity will be extended farther than anticipated.

But this notion of similarity can not appear clearly unless the influence of the different vortices is separated. This means changing from the correlation functions to spectral functions.

It will be remembered (compare sections 11 and 12) that to the spatial correlation tensor $R_{\alpha\beta}(\xi) = \overline{u_\alpha u'_\beta}$ there corresponds the spectral tensor $\varphi_{\alpha\beta}(\lambda)$ defined by

$$R_{\alpha\beta}(\xi) = \int_\Lambda \varphi_{\alpha\beta}(\lambda) e^{i \sum_{p=1}^{3} \lambda_p \xi_p} \, d\lambda$$

If there is isotropy, the spectral tensor depends on a single spectral function $F(k)$, just as the spatial correlation tensor depends on a single function $R(r)$ for example.

To the properties of $R(r)$ for small values of r correspond those of $F(k)$ for great values of k, as results from formula (14-4) which links F to R, and which is recalled here:

$$R(r) = 2 \int_0^\infty \frac{\sin rk}{rk} F(k) \, dk$$

It has been shown (section 29) that to the law of correlation in $r^{\frac{2}{3}}$ for $R(r)$, applicable in the domain $l \ll r \ll L$, and resulting from Kolmogoroff's second hypothesis, there corresponds a spectral law of the form

$$F(k) = A\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$$

where A is a universal constant. This law is valid in the domain $k_0 < k < k_s$ and the numbers lk_p, Lk_0 are of the order of unity.

In the most general case of Kolmogoroff's first hypothesis, one can only put:

$$F(k) = F_0\Phi\left(\frac{k}{k_0}\right)$$

for $k > k_0$, F_0 and k_0 being functions of ϵ and ν only and Φ a universal function.

Putting $\int_0^k \Psi(k')dk' = \underline{T}(t)$ in (22-8), gives

$$\frac{\partial}{\partial t}\int_0^k F(k')dk' = -\underline{T}(k) - \nu\int_0^k 2k'^2F(k')dk' \qquad (31\text{-}1)$$

The function $\underline{T}(k)$ represents the transfer of energy by turbulence of the large eddies to smaller eddies.

In the case in which the dissipation by viscosity is negligible, the conditions of statistical equilibrium are written simply $\underline{T}(k) = c^{te}$. A dimensional analysis affords the spectral law in $k^{-\frac{3}{5}}$. In fact, it can be said that $\underline{T}(k)$ has the dimensions of averages so that $u_1u_2\frac{\partial u_1}{\partial x}$ is

$v_k^3 k$, where v_k is a certain characteristic velocity of vortices having k for wave number. Hence one must have $v_k^3 k = c^{te}$, in the domain $k_s \ll k \ll k_0$, and v_k is proportional to $k^{-\frac{1}{3}}$.

For velocity v_k, the quantity $\sqrt{\sum \overline{(u'_i - u_i)^2}}$ which, according to Kolmogoroff's theory, is proportional to $r^{\frac{2}{3}}$ in the chosen conditions, can be assumed. Weizsäcker's law is equivalent to Kolmogoroff's law because $F(k)$, having the dimensions of $\frac{v_k^2}{k}$, is proportional to $k^{-\frac{5}{3}}$. (Compare section 25.)

The variation with k of other quantities associated with different vortices can also be determined. For example, the characteristic period of the vortices is proportional to $\frac{1}{kv_k}$, a quantity which has the dimensions of a time interval, that is, to $k^{-\frac{2}{3}}$. It decreases when k increases, which is in agreement with the hypothesis according to which the characteristic periods of development of the small vortices are shorter than those of large size vortices.

It can likewise be verified that the viscous dissipation, which is proportional to $k^2 F$, varies like $k^{\frac{1}{3}}$. Hence, it cannot be neglected for great values of k.

If now the viscosity is no longer neglected, the form of the function $\underline{T}(k)$ must be specified. This is what Heisenberg did who, as shown in section 25, has assumed that $\underline{T}(k)$ could be put in the same form as the losses of energy by viscosity, the form of the "turbulent viscosity" coefficient resulting from similarity considerations.

The comparison with experiment of the spectral law $k^{\frac{5}{3}}$ has not as yet been carried out on flows in which the Reynolds number is high enough so that the conclusions are well defined. Nevertheless, its range of validity seems more extended than that of the law $r^{\frac{2}{3}}$ for the correlations, because the influence of the various vortex sizes is different due to the use of the spectral analysis.

Townsend attempted to check the validity of the expression of the function $\underline{T}(k)$ given by Heisenberg, and according to which F should be proportional to k^{-7} at great values of k. The mean value of $\overline{\left(\frac{\partial^3 u_1}{\partial x_1^3}\right)^2}$ is proportional to $\int_0^\infty k^6 F(k)dk$. If the law in k^{-7} were verified, this integral should be infinite, which signifies that the integral $\int_{k_1}^{k_2} k^6 F(k)dk$ should increase considerably together with its upper limit k_2. However, the experiment seems to indicate that this integral has a limiting value independent of k_2, and variable during the decay of turbulence. Therefore, the law in k^{-7} cannot be true for the highest values of k, which limits the range of validity of Heisenberg's theory.

CHAPTER V

DECAY OF THE TURBULENCE BEHIND A GRID

32. History:

The theory of local isotropy shows that the laws of turbulent flows are the same, regardless of the origin of turbulence, provided that the conditions are "local" and the Reynolds numbers high enough. This chapter deals with the particular laws of a turbulent flow of great importance, that which is produced in the test chamber of wind tunnels, and in which turbulence is largely due to a grid, that is, an obstacle of periodic structure. Turbulence arises from the mixing of wakes of grid elements, and is dissipated progressively as it moves farther away from the grid. For an observer carried along with the velocity of the ensemble, the turbulence is therefore subjected to a "decay" as function of the time. The laws of this decay are, naturally, compatible with the local laws of Kolmogoroff, but today they are known with a high degree of accuracy in a much more extended domain, and it can be stated that it is in the study of this phenomenon that the theory of turbulence, steadily checked by experiment, has made the fastest and most constructive advance.

The basis of this theory was established by G. I. Taylor, who, by considerations of similarity, had indicated that, if the Reynolds number is high, the turbulent energy $\frac{1}{2}u_0^2$ must decrease as t^{-1} as a function of the time.

The measurements made since by numerous researchers have shown that this law is close to experimental results; nevertheless the experimental curves were rather of the type

$$u_0^2 t^n = C^{te}$$

n ranging between 1 and 2. As the precision of the measurement is low as soon as the turbulence intensity becomes low, the interval to which the comparisons refer, is restricted.

The variation of the length of dissipation λ in terms of time is tied directly to that of the intensity of turbulence by the relation (19-1)

$$\frac{1}{u_0^2}\frac{du_0^2}{dt} = -\frac{10\nu}{\lambda^2}$$

If u_0^2 varies as t^{-n}, one deduces from it:

$$\lambda^2 = \frac{10\nu}{n}t$$

But, up to the last few years, the measurements of λ offered no possibility of defining the value of n. The value of λ is, in effect, deduced from the curvature at the origin of the curve representing the correlation function $f(r)$. The measurements are made with two anemometers, of which the distance r cannot be reduced indefinitely for manifold reasons (geometrical and physical). As a result, the curve $f(r)$ is not well known for small values of r.

However, an examination of the correlation curves seems to indicate that their shape remains constant in terms of the distance from the grill, during the decay of turbulence.

All these questions have taken a decisive forward step, as a result of A. A. Townsend's new methods which enable direct measurement, by electrical

methods, of the momentary or mean values of the quantities u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and those of the products of two or three quantities, as well as the direct determination of the statistical frequency curve of a quantity in terms of the velocity.

The use of these methods enabled G. K. Batchelor and A. A. Townsend to determine the law of decay of turbulence produced by a grill with a good degree of accuracy over a wide range of values of the Reynolds numbers of the grill, defined by

$$R_M = \frac{UM}{\nu} \tag{32-1}$$

where M is the width of the mesh, U the overall velocity.

When the turbulence is isotropic

$$\frac{d}{dt}u_0^2 = -10\nu\frac{u_0^2}{\lambda^2} = -10\nu\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \tag{32-2}$$

The direct measurement of $\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}$ affords the evaluation of λ much more quickly and more accurately than by the procedure employed previously, and consequently defines λ in terms of time, that is, of the distance from the turbulence-producing grid.

33. Initial and final phase of turbulence (Batchelor and Townsend, references 8 and 11):

Townsend's measurements made it possible to distinguish in the decay of turbulence an initial phase and a final phase, separated by a less well defined intermediary phase.

The initial phase, clearly proved by these experiments, corresponds to grill distances of less than 100 or 150 mesh widths (variable according to the Reynolds number). It is characterized by the fact that the decay of turbulent energy can be represented by an expression proportional to t^{-n} where the exponent n differs from unity by less than 10 percent. It is said that u_0^2 decreases as t^{-1}.

In the final phase, which corresponds to the limiting state of decay of turbulence, experiments prove that u_0^2 decreases as $t^{-\frac{5}{2}}$.

The initial phase is also characterized by the fact that the Reynolds number of turbulence, defined by

$$R_\lambda = \frac{u_0 \lambda}{\nu} \tag{33-1}$$

remains constant throughout the period of decay to which it corresponds.

If, in fact, $u_0^2 = At^{-n}$, it is seen that $\lambda^2 = \frac{10\nu}{n}t$, and that

$$R_\lambda = \sqrt{\frac{10A\nu}{n}}\, t^{\frac{1-n}{2}}$$

Thus R_λ remains constant if $n = 1$, and in this case only.

To define the structure of the two essential phases of turbulence, we now introduce the rotation (Batchelor and Townsend, reference 10) of which the relations with the correlation functions and the dissipation length λ are simple, at least when the turbulence is isotropic. The general equations relevant to the rotation are simplified by the Kármán-Howarth equation.

The rotation of the velocity, defined by the formulas

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \qquad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \qquad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{33-2}$$

satisfies two well known equations which are obtained by simple combinations of the equations of motion

$$\frac{\partial u_i}{\partial t} + \sum_l u_l \frac{\partial u_i}{\partial x_l} = \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i \tag{33-3}$$

If (33-3) is derived with respect to x_i, since one interchanges i and j and subtracts the two equations obtained, the result is

$$\frac{\partial \omega_i}{\partial t} + \sum_k u_k \frac{\partial \omega_i}{\partial x_k} = \sum_k \omega_k \frac{\partial u_k}{\partial x_i} + \nu \Delta \omega_i \qquad (33\text{-}4)$$

The two members of (33-4) are multiplied by $2\omega_i$, or added with respect to i, with due allowance for incompressibility, the averages taken and homogeneous turbulence assumed. Putting $3\overline{\omega^2} = \sum \overline{\omega_i^2}$ leaves

$$3\frac{\partial}{\partial t}\overline{\omega^2} = 2\sum_{ik} \overline{\omega_i \omega_k \frac{\partial u_i}{\partial x_k}} + 2\nu \sum_{ik} \overline{\omega_i \frac{\partial^2 \omega_i}{\partial x_k^2}} \qquad (33\text{-}5)$$

If, in addition, the turbulence is isotropic, then (section 9):

$$\overline{\omega^2} = \overline{\omega_1^2} = \overline{\omega_2^2} = \overline{\omega_3^3} = 5\frac{u_0^2}{\lambda^2}$$

In this equation, the first term of the second member represents the increment of the vorticity due to the elongation of the vortex filaments, and the second the decrement due to the influence of the viscosity. It is verified that this last term really is negative because

$$2\overline{\omega_i \frac{\partial^2 \omega_i}{\partial x_k^2}} = \frac{\partial^2 \overline{\omega_i^2}}{\partial x_k^2} - 2\overline{\left(\frac{\partial \omega_i}{\partial x_k}\right)^2}$$

and $\dfrac{\partial^2 \overline{\omega_i^2}}{\partial x_k^2}$ is zero if the turbulance is homogeneous.

Batchelor and Townsend measured the three terms of (33-5) directly. But it is convenient to give equation (33-5) a slightly different and easier-to-interpret aspect. The Kármán-Howarth equation (18-5) can be

used to evaluate the first member. Introducing the limited developments of f, c, up to the terms in r^4 (cf. section 9)

$$f(r) = 1 + \frac{r^2}{2}f''(0) + \frac{r^4}{24}f^{(4)}(0) + \ldots$$

$$c(r) = r^3\,\frac{c'''(0)}{6} + \ldots$$

and identifying the terms in r^2, gives

$$\frac{\partial}{\partial t}\left[u_0{}^2 f'''(0)\right] = \frac{14}{3}\nu u_0{}^2 f^{(4)}(0) + \frac{7}{3}u_0{}^3 c'''(0) \qquad (33\text{-}6)$$

Now, the derivatives of the functions f and c can be expressed in terms of the average values bearing on the velocity derivatives. The calculation, as done before for the rotation, gives

$$u_0{}^2 f''(0) = -\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} \qquad u_0{}^2 f^{(4)}(0) = \overline{\left(\frac{\partial^2 u_1}{\partial x_1{}^2}\right)^2} \qquad u_0{}^3 c'''(0) = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}$$

(33-7)

If therefore the nondimensional quantities

$$S = -\frac{\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}}{\left[\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}\right]^{\frac{3}{2}}} \qquad G = u_0{}^2\frac{\overline{\left(\frac{\partial^2 u_1}{\partial x_1{}^2}\right)^2}}{\left[\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}\right]^2} \qquad (33\text{-}8)$$

are introduced of which the first has already played a prominent part in section 28 (coefficient of dissymmetry, for $r = 0$), it is seen that

$$S = -\frac{c'''(0)}{\left[-f''(0)\right]^{\frac{3}{2}}} = -\lambda^3 c'''(0) \qquad \underline{G} = \frac{f^{(4)}(0)}{f''^2(0)} = \lambda^4 f^{(4)}(0) \tag{33-9}$$

λ being, as will be remembered, a "length of dissipation" defined by $f''(0) = -\frac{1}{\lambda^2}$.

Since $\overline{\omega^2} = 5\frac{u_0^2}{\lambda^2}$, equation (33-6) can be written

$$\frac{\partial \overline{\omega^2}}{\partial t} = \frac{7}{3\sqrt{5}}\left(\overline{\omega^2}\right)^{\frac{3}{2}}\left(S - \frac{2\underline{G}}{\underline{R}_\lambda}\right) \tag{33-10}$$

where $\underline{R}_\lambda = \frac{u_0 \lambda}{\nu}$ is the Reynolds number of turbulence already defined by equation (33-2).

So in turbulence language, equation (33-10) is a consequence of the Kármán-Howarth equation of turbulence, equivalent to (33-5), but superior for comparisons with experiments.

The measurements of the various terms of (33-10) during the initial period of decay of turbulence confirm, first, that

$$\underline{R}_\lambda = C^{te} \qquad \lambda^2 = 10\nu t \tag{33-11}$$

t being counted from any convenient origin.

Consequently one may write:

$$\overline{\omega^2} = 5\frac{u_0^2}{\lambda^2} = \frac{1}{20}\frac{\underline{R}_\lambda^2}{t^2} \tag{33-12}$$

and (33-10) takes the form

$$\underline{G} = \frac{1}{2}\underline{R}_\lambda S + \frac{30}{7} \qquad (33\text{-}13)$$

It has been shown (section 28) that the measurements of the coefficient of asymmetry S prove that the latter is independent of $\underline{R}_\lambda$ and close to 0.38. The direct measurements of G are in good agreement with the above formula, in which S = 0.38, which constitutes a check on the qualities of the measurements.

These results make it possible to evaluate the ratio of the two terms of the second member of the equations (33-5) or (33-10). The quotient of the term representing the dissipation through viscosity to that which expresses the inertia effects is equal to

$$\frac{2\underline{G}}{S\underline{R}_\lambda} = 1 + \frac{60}{7S\underline{R}_\lambda} \qquad (33\text{-}14)$$

or, replacing S, by its experimental value

$$1 + \frac{22}{\underline{R}_\lambda} \qquad (33\text{-}15)$$

This ratio is constant throughout the initial period of decay of turbulence. If $\frac{22}{\underline{R}_\lambda}$ is considerably less than unity, the contributions to $\frac{\partial \overline{\omega^2}}{\partial t}$ of the viscosity and of the inertia terms are of comparable importance and the velocity resulting from the variation of the vorticity is of an order of magnitude lower than each one of these two terms. This is the range of high Reynolds numbers $\underline{R}_\lambda$, where the theory of statistical equilibrium (Kolmogoroff, Weizsäcker) applies. For large enough values of the wave number k, the spectral law $k^{-\frac{5}{3}}$ can be observed there. The

measurements show, in fact, that, in proportion as R_λ increases, the correlation curves approach the limiting shape with tangent vertical to the origin as implied by the formula

$$f = 1 - cr^{\frac{2}{3}}$$

The original parabolical region becomes consistently smaller as R_λ becomes higher (Figure 11; compare also the end of section 25).

If $\frac{22}{R_\lambda}$ approaches unity (Reynolds numbers neither too high nor too low), the results deteriorate and new partial or total similarity hypotheses must be made, resulting in solutions of the type of those studied in section 21 from the purely mathematical point of view, and which are not repeated here.

If $\frac{22}{R_\lambda}$ is great (low Reynolds numbers), the effect of the triple correlations is negligible against the viscous dissipation. The mechanism of the decay of turbulence is mainly due to the viscosity, and the solutions of section 20 can be invoked. That is what happens in particular in the final phase of decay which is discussed below.

34. Concepts regarding the structure of the final phase of turbulence (Batchelor and Townsend, reference 12; Batchelor, reference 8):

It is necessary to predict by theory the law of the final phase of decay of turbulence, according to which u_0^2 decreases as $t^{-\frac{5}{2}}$, and, consequently

$$\lambda^2 = 4\nu t \qquad (34\text{-}1)$$

When the turbulence is sufficiently attenuated, the fluctuation is slight, and the inertia effects negligible against the effects of viscosity. As a result, the nonlinear terms in the equations of motion can be discounted, and one may simply write

$$\frac{\partial u_i}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i$$

In these conditions the equations of motion can be integrated, the velocity components computed in terms of time and the initial conditions, and the correlations deduced. This is the program set forth in section 16, and it cannot be accomplished except in very special cases. Thus the time correlations can be calculated just as well as the classical space correlations.

Only the case of isotropic turbulence is analyzed. To neglect the inertia terms is to neglect the triple correlations in the "fundamental equations." Thus one reverts to the conditions of section 20, or turns to the possibility of expressing the spectral equation in the simplified form

$$\frac{\partial F}{\partial t} + 2\nu k^2 F = 0 \qquad (34\text{-}2)$$

The general solution of this equation reads

$$F(k,t) = F\left(k,t_1\right)e^{-2\nu k^2\left(t-t_1\right)} \qquad (34\text{-}3)$$

t_1 being the instant that makes the start of the final phase.

One may also use equation (20-1), equivalent in correlation terms to the spectral equation (34-2), which, with the triple correlations disregarded, reads

$$\frac{\partial R}{\partial t} = 2\nu\,\Delta R$$

of which the general solution (20-9) is a little harder to write.

But these solutions must have an asymptotic character. It is of no interest here, except when $t - t_1$ is sufficiently great. $F(k,t)$ approaches zero when $t - t_1$ approaches infinity, as is natural, since the turbulence decays progressively. So, when $t - t_1$ is great, the only regions of the spectrum continuing to exist are those for which k is small, $k^2\left(t - t_1\right)$ remaining practically finite. It has been seen (in section 18) that, for small values of k, $F(k)$ was necessarily of the form Ck^4, C being a constant independent of the time, and tied to the Loïtsiansky invariant.

Consequently, for the large values of t, the spectral function takes the form

$$F(k,t) = Ck^4 e^{-2\nu k^2 (t-t_1)} \tag{34-4}$$

valid for small values of k, that is, for large-size vortices.

Example 1 in section 20 gave the corresponding form of the correlation functions $R(r,t)$ and $f(r,t)$. In particular it is recalled that

$$f(r,t) = e^{-\frac{r^2}{8\nu(t-t_1)}}$$

and that

$$u_0^2(t) = \frac{2}{3}\int_0^\infty F(k)dk = \frac{3}{32}\frac{\sqrt{\pi}}{\sqrt{2}}C\left[\nu\left(t - t_1\right)\right]^{-\frac{5}{2}}$$

This is the law of decay that had to be explained. The proof rests on the hypotheses of section 20 and section 18, particularly on the fact that the Loïtsiansky invariant is finite and other than zero. The agreement of theory and experiment confirms them.

In this calculation, it was supposed that the turbulence was isotropic. On the other hand, only the large size vortices continued to exist because the energy contained in the region of the spectrum corresponding to substantial values of k becomes negligible. As the constant C is independent of time, the shape of the spectrum for small values of k is independent of the state of decay of turbulence. The large-size vortices contain very little energy, but, in the final phase, the energy of the small vortices is dissipated and only the large vortices remain visible whose energy, as feeble as it is, has become very great proportionally.

Now, while the turbulence is always locally isotropic and exhibits a tendency to isotropy for the small vortices (large values of k), it is no longer the same for the large-size vortices, in which the geometrical dissymmetries of the wind tunnel are exhibited.

A resumption of the calculations, but without introducing the hypothesis of isotropy, produces similar conclusions; the portion of the spectrum (that is, of the spectral tensor) relative to small values of k remains constant during the decay, while the rest of the spectrum vanishes progressively. It follows that the anisotropy of turbulence that results from the geometrical and mechanical configuration of the flow, and which is practically concealed at the beginning of the decay of turbulence, must appear in the final phase (Batchelor, ref. 9). Unfortunately, it is rather difficult to check it, since the turbulence is then very weak.

35. The concept of "dynamic statistical equilibrium" (Heisenberg, ref. 23, Batchelor, ref. 8):

In the problem of the decay of turbulence behind a grill, the study of the shapes of spectral curves and correlation curves led Heisenberg to believe that the notion of similarity could be extended to vortices containing the major part of the turbulent energy. The overall spectrum, with exception of the region of small wave numbers or large vortices (spectrum in k^4), thus would be in a dynamic statistical equilibrium or quasi-equilibrium, in which the energy distribution in the spectrum is modified during the decay of turbulence, the maximum shifting toward the small wave numbers, but the general shape of the curve remaining the same.

According to the static theory of local isotropy, the spectral function $F(k)$ can be written in the form

$$F(k) = F_0 \Phi\left(\frac{k}{k_s}\right) \tag{35-1}$$

where $\frac{1}{k_s}$ is a length that fixes the transition between the spectral region of predominant inertia forces and that where the viscosity forces become comparable to them. Φ is a nondimensional function. F_0 and k are dependent on ϵ and ν only, and dimensional analysis shows that

$$F_0 = \nu^{\frac{5}{4}}\epsilon^{\frac{1}{4}} \qquad k_s = \nu^{-\frac{3}{4}}\epsilon^{\frac{1}{4}} \tag{35-2}$$

It was shown (in section 29) that, when the viscosity is negligible, F becomes proportional to $k^{-\frac{5}{3}}$, which, in this limiting case, fixes

the form of the function Φ. In the general case, the form of the function Φ depends upon that of the function $\Psi(k)$ or $\underline{T}(k)$ which appears in the second term of the fundamental equation (22-8) or (31-1), and which represents the transfer of energy through turbulence from the large to the small size vortices. Φ is in all cases a universal function of $\frac{k}{k_s}$, and the similarity law which it defines is valid in a domain $k_0 < k < k_s$, where k_0 is an approximate lower limit. This domain is the more extended the higher the Reynolds number $\underline{R}_0 = \frac{u_0}{k_0 \nu}$. This number characterizes the importance of the portion of the spectrum in absolute equilibrium. This portion is maintained during the decay of turbulence, so that $\underline{R}_0$ remains constant, as well as the nondimensional ratio $\frac{u_0^2}{\sqrt{\epsilon \nu}}$. It is found that this number intervenes when the region of absolute equilibrium is left. The quasi-equilibrium therefore depends on the parameters ϵ and ν, but, in addition, on a third parameter, the total energy $E = \frac{3}{2} u_0^2$, of which the dissipation ϵ is the derivative with respect to time. By (19-1) and (19-2), $\frac{u_0^2}{\sqrt{\epsilon \nu}}$ is proportional to the Reynolds number $\underline{R}_\lambda = \frac{u_0 \lambda}{\nu}$ of turbulence. For a quasi-equilibrium to be possible, $\underline{R}_\lambda$ must therefore be constant during the development of turbulence. This is what the experiment proves during the initial phase of decay of turbulence, and this result supplies an argument in favor of the quasi-equilibrium hypothesis. If $\underline{R}_\lambda$ varies, dynamic similarity ceases, but that does not prevent the possible existence of a region of local isotropy, variable during the development.

By virtue of the known formula (14-11)

$$\lambda^2 = 15 \nu \frac{u_0^2}{\epsilon}$$

The parameters ν, ϵ and u_0^2 can be replaced by ν, λ and u_0^2. If $\underline{R}_\lambda = \frac{u_0 \lambda}{\nu}$ is constant in a defined regime of decay, the spectral function $F(k,t)$ depends only on a variable parameter, λ for example, equivalent to the time, and in addition, proportional to $\sqrt{t}$.

We write

$$F(k,t) = F(k_1,t)\Phi(\eta) \tag{35-3}$$

where $\eta = \frac{k}{k_1}$ is a nondimensional parameter dependent on the fixed wave number k_1. The nondimensional function Φ is completely defined by the value of R_λ, that is, by the initial conditions at the grid. The length of dissipation λ can serve as reference length $\frac{1}{k_1}$. The dimensional analysis proves then that $F(k_1,t)$ is the product resulting from $u_0^2\lambda$, or $\frac{R_\lambda^2}{\lambda}$, a quantity proportional to $\frac{1}{\sqrt{t}}$ multiplied by a quantity constant during the decay. Introducing a velocity v_0 and a reference length $\frac{1}{k_0}$, one may put:

$$\eta = k\lambda = k\sqrt{\frac{v_0 t}{k_0}} \qquad F(k,t) = \frac{1}{\sqrt{t}}\left(\frac{v_0}{k_0}\right)^{\frac{3}{2}}\Phi(\eta) \tag{35-4}$$

The form of the function Φ is now defined by means of the fundamental spectral equation (22-8), which we recall here:

$$\frac{\partial F}{\partial t} + 2\nu k^2 F = \Psi \tag{35-5}$$

We examine first what happens in case of small values of parameter η. We shall show that $\Phi(\eta)$ is, according to (35-4), infinitely small of the first order with respect to η, and that the position of the tangent at the origin to the spectral curves $F(k,t)$ is independent of t. For the sake of simplification

$$F(k,t) = \frac{1}{\sqrt{t}}\Phi_1(\eta_1) \qquad \eta_1 = k\sqrt{t}$$

By (35-5)

$$\eta_1\Phi'_1 - \Phi_1 + 4\nu\eta_1^2\Phi_1 = 2t^{\frac{3}{2}}\Psi \qquad (35\text{-}6)$$

Assuming that Φ_1 allows a development of the form

$$\Phi_1(\eta_1) = \alpha_0\eta_1^{\,n} + \alpha_1\eta_1^{\,n+1} + \alpha_2\eta_1^{\,n+2} + \ldots$$

the first member of (35-6) becomes, except for the factor $t^{-\frac{3}{2}}$,

$$(n-1)\alpha_0\eta_1^{\,n} + n\alpha_1\eta_1^{\,n+1} + \left[(n+1)\alpha_2 + 4\nu\alpha_0\right]\eta^{n+2} + \ldots \qquad (35\text{-}7)$$

It was seen in section 22, that $\Psi(k)$ was infinitely small as k^6, that is, as η^6. It follows that, either

$$n = 1 \qquad \alpha_1 = 0 \qquad (n+1)\alpha_2 + 4\nu\alpha_0 = 0, \ldots$$

where the first term of (35-7) must be the term in η^6; or else:

$$n = 6$$

In the hypothesis $n = 6$, there would result, for $F(k,t)$ a law k^6 which is incompatible with the exact law k^4 and with experiment.

For $n = 1$, there is for $F(k_1t)$ a law in k which is not compatible with experiment when $k \longrightarrow 0$ but is verified for small enough values of k, except at the limit. In that case

$$\Phi_1(\eta) = \alpha_0\eta_1 + \alpha_2\eta_1^3 + \ldots$$

and

$$F(k,t) = \alpha_0 k + \alpha_2 t k^3 + \ldots \tag{35-8}$$

α_0, α_2 being constants independent of t. It follows that

$$\frac{\partial F}{\partial k} = \alpha_0 + 3\alpha_2 t k^2 + \ldots$$

approaches a limit independent of t when $k \longrightarrow 0$. All spectral curves compatible with the quasi-equilibrium have the same tangent at the origin.

Therefore, if the part of the spectrum in k^4 (represented by dotted curves in figure 12) is neglected, the spectral curves are approximately homothetic with respect to the origin, the maximum decreasing and shifting toward the small values of k when the time increases, that is, with progressing decay of turbulence.

The similarity is obtained again on the correlation curves f and g. In fact, (14-4)

$$R(r) = 2\int_0^\infty \frac{\sin rk}{rk} F(k,t)\,dk = \frac{2}{t}\int_0^\infty \frac{\sin \frac{r}{\sqrt{t}}\eta_1}{\frac{r}{\sqrt{t}}\eta_1} \Phi_1(\eta_1)\,d\eta_1 \tag{35-9}$$

The functions Rt, and hence $\frac{R}{u_0^2}$, f, g, depend therefore on the variable $\frac{r}{\sqrt{t}}$ and not separately on r and t. In other words, the shape of the correlation curves is maintained, provided that $\frac{r}{\sqrt{t}}$ or $\frac{r}{\lambda}$ is taken as variable.

Incidentally, $\frac{r}{L}$ can be taken as variable also, because

$$L = \int_0^\infty f\,dr = \frac{1}{2u_0^2}\int_0^\infty R\,dr = \frac{\pi}{2u_0^2}\int_0^\infty \frac{F(k)}{k}\,dk = \frac{\pi}{2u_0^2}\int_0^\infty \frac{\Phi_1(\eta_1)}{\eta_1}\,\frac{d\eta_1}{\sqrt{t}} \tag{35-10}$$

Since in the region where $\underline{R}_\lambda = C^{te}$, u_0^2 is inversely proportional to t, L is proportional to $\sqrt{t}$ as λ, and $\frac{L}{\lambda}$ is constant.

But the similarity with L is more quickly destroyed during the decay of turbulence than the similarity with λ, because in the expression of L, the values of Φ_1 corresponding to the small values of η_1 are of greater weight than the others. The law $u_0^2\lambda^2 = c^{te}$ prevails therefore much longer than the law $u_0^2L^2 = c^{te}$. The explanation is that similarity is approached with respect to L, while it still seems to be verified for λ. It can be seen that the similarity discrepancies of the correlation curves appear, first, for the great values of r, and that they afterward reach the central zone. The reason for this zone to start changing is that the maximum energy of the spectral curve reaches the spectral region in k^4. The law of decrease of energy can no longer be of the form $u_0^2t = C^{te}$. The turbulence reaches the final period of decay, in which the energy u_0^2t decreases as $t^{-\frac{5}{2}}$.

In this whole analysis it is not necessary to assume that $\underline{R}_\lambda$ is high. The main point is that the initial conditions are such that the region, in which the spectral function F varies as k^4, be relatively small. If $\underline{R}_\lambda$ is very high, there exists a region of the spectral curve in which F varies as $k^{-\frac{5}{3}}$. This region does not exist when $\underline{R}_\lambda$ is medium.

If it is desired to go farther and define the form of the function F, the form of the energy transfer $\Psi(k)$ (Heisenberg, reference 24) must be

chosen. If $\Psi(k)$ has the form given to it by Heisenberg (section 25), for instance

$$\underline{T}(k) = \int_0^k \Psi(k')dk' = 2C \int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\, dk' \int_0^k k'^2 F(k')dk' \qquad (35\text{-}11)$$

the reduced variables (35-4) are employed, and it is proved that (35-5) leads to the reduced equation

$$\int_0^\eta \Phi(\eta')d\eta' - \frac{1}{2}\eta\Phi(\eta) = 2\left[A + C\int_\eta^\infty \sqrt{\frac{\Phi(\eta')}{\eta'^3}}\, d\eta'\right] \int_0^\eta \eta'^2\Phi(\eta')d\eta' \qquad (35\text{-}12)$$

where $A = \nu\frac{k_0}{v_0}$ is the inverse of a Reynolds number.

Equation (35-12) is easily reduced to a differential equation which, contrary to what occurs in the case of the steadiness hypothesis (25-6), seems unsolvable. Heisenberg (reference 24) has pointed out approximate solutions, valid for great or small values of A. (The reader is referred to Heisenberg's report and also to Batchelor, reference 8).

36. Synthesis of the results relating to the structure of the spectrum of turbulence:

Obviously only isotropic turbulence is involved. The results collected regarding Heisenberg's spectral function $F(k)$ which, incidentally, cannot be measured directly, but is deduced, by simple transformations, from the directly measurable spectral functions (spectral functions of Taylor, insofar as they define a spectral distribution in space, and not in time) or from the correlation functions.

Whatever the structure of turbulence may be, the spectral laws tend, when the Reynolds number becomes high enough, toward an absolute limiting law, where $F(k)$ is proportional to $k^{-\frac{5}{3}}$ (absolute equilibrium). The corresponding correlation functions are of the form $f(r) = 1 - C^{te}r^{\frac{2}{3}}$.

In the problem of the decay of turbulence behind a grid, an initial, an intermediate and a final period are distinguished. For small values of k, $F(k)$ is always $= Ck^4$, C being a constant independent of time (bound to the invariant of Loïtsiansky).

For the average values of k, and in the initial period, a state of "dynamic equilibrium" exists (figure 13).

The corresponding portions of the spectral curves are homothetic with respect to the origin and, when the time of decay increases, their maximum approaches the origin. Their shape is known up to the origin, but the straight part that corresponds to the small values of k and which is fixed during the decay of turbulence, has no physical reality and must be replaced by the part of the spectrum in k^4.

If the Reynolds number is high enough, the portion of the spectrum corresponding to great values of k is defined by the laws of absolute equilibrium $\left(\text{spectrum in } k^{-\frac{5}{3}}\right)$. When $k \longrightarrow \infty$ the spectrum terminates in a curve k^{-7} (experimentally doubtful). The region $k^{-\frac{5}{3}}$ disappears if the Reynolds number is not very high.

If the Reynolds number is small enough, the region in dynamical equilibrium is numerically known: $F(k)$ is proportional to $ke^{-2k^2\nu t}$, provided that $k^2\nu t$ is not too great.

In this initial period the energy of the fluctuations varies as t^{-1}.

In the final period the zone of dynamic similarity disappears, and the maximum of the spectral curve intrudes on the region k^4 (figure 14). Starting from an instant t_1 marking the beginning of this phase we can write

$$F(k) = Ck^4 e^{-2\nu k^2 (t-t_1)}$$

and the energy of the fluctuations decreases as $\left(t - t_1\right)^{-\frac{5}{2}}$.

Index of Principal Notations, Fundamental Formulas, and Dimensional Equations

Certain letters have been employed successively, in various paragraphs, to represent different quantities without having to be afraid of confusion. Others, on the contrary, always represent the same quantity; at least from chapter II on. We shall enumerate them and recall the principal formulas into which they enter and to which one has frequently to refer:

x_1, x_2, x_3 represent the coordinates of a point M

ξ_1, ξ_2, ξ_3 designate the differences of the coordinates of two points M, M', of which the distance is r. (In the first chapter, through a few lines, r represents, by exception, a correlation coefficient.)

$f(r), g(r)$ designate Kármán's double correlation functions. (In chapter I, f is used to represent certain statistical frequencies and probability densities.)

These functions are derived from the correlation tensor $R_{\alpha\beta}(\xi_1, \xi_2, \xi_3, t)$ in space, of which $R = \sum_{\alpha} R_{\alpha\alpha}$ is the scalar invariant. (In the first chapter, R denotes the density of probability of position and velocity of a statistical point.) R should not be confused with $\underline{R}$ which denotes a Reynolds number particularized by a subscript: $\underline{R}_\lambda$, $\underline{R}_M$, . . .

$a(r), b(r), c(r)$ denote the triple correlation functions of Kármán (section 9) which are deduced from the triple-correlation tensor $T_{\alpha\beta\gamma}$

$\lambda_1, \lambda_2, \lambda_3$ are the vector components of the wave number (spatial frequency) in the space Λ of the wave numbers

k is the length of the vector λ_i

F(k) is Heisenberg's spectral function, associated with the spectral tensor $\varphi_{\alpha\beta}(\lambda_1, \lambda_2, \lambda_3)$

E denotes the total energy of the fluctuation of turbulence per unit mass

ϵ denotes the energy dissipated as heat per unit mass

ν denotes the coefficient of kinematic viscosity

The letter u always designates a velocity. From chapter II on u_1,u_2,u_3 represent the components of the velocity of the fluctuations at the point M, and, if the turbulence is isotropic, u_0 is the mean square value of u_i.

λ is the dissipation length, L the correlation length

h always represents a statistical function with orthogonal increments

The quantities enumerated above have the following dimensional equations:

$$k,\lambda_1,\lambda_2,\lambda_3 = \underline{L}^{-1}$$

$$F(k) = \underline{L}^3\underline{T}^{-2}$$

$$E = \underline{L}^2\underline{T}^{-2}$$

$$\nu = \underline{L}^2\underline{T}^{-1}$$

$$R = \underline{L}^2\underline{T}^{-2}$$

$$\epsilon = \underline{L}^2\underline{T}^{-3}$$

By way of comparison, the dimensional equations of classical mechanical quantities read:

$$\text{Velocity} = \underline{LT}^{-1}$$

$$\text{Acceleration} = \underline{LT}^{-2}$$

$$\text{Force} = \underline{MLT}^{-2}$$

$$\text{Work} = \underline{ML}^2\underline{T}^{-2}$$

$$\text{Power} = \underline{ML}^2\underline{T}^{-3}$$

There follows a list of the principal formulas, referred to constantly in chapters IV and V. The sections in which these formulas appear for the first time are shown in parentheses.

(Section 8)

$$R_{\alpha\beta}(\xi_1,\xi_2,\xi_3,t) = \overline{u_\alpha(x)u_\beta(x')}$$

$$= u_0^2\left[\frac{f - g}{r^2}\xi_\alpha\xi_\beta + g\delta_{\alpha\beta}\right] \quad \text{(isotropic turbulence)}$$

$$R = \sum_{\alpha=1}^{3} R_{\alpha\alpha}$$

$$T_{\alpha\beta\gamma} = \overline{u_\alpha(x)u_\beta(x)u_\gamma(x + \xi)}$$

(Section 9)

$$\lambda = \sqrt{-\frac{2}{g''(0)}} \qquad L = \int_0^\infty f(r)dr$$

$$g = f + \frac{r}{2}\frac{\partial f}{\partial r} \quad \text{(incompressibility)}$$

(Section 13)

$$E = \frac{3}{2}u_0^2 = \int_0^\infty F(k)dk$$

$$\epsilon = 2\nu\int_0^\infty k^2F(k)dk$$

(Section 14)

$$R(r) = 2\int_0^\infty \frac{\sin rk}{rk}F(k)dk$$

$$F(k) = \frac{1}{\pi}\int_0^\infty rk \sin rk\, R(r)dr$$

$$\lambda^2 = 10\nu\frac{E}{\epsilon}$$

(Section 19) $\frac{\partial E}{\partial t} = -\epsilon \qquad \frac{d}{dt}u_0^2 = -10\nu\frac{u_0^2}{\lambda^2}$

(Section 22)
$$\begin{cases} \frac{\partial F}{\partial t} + 2\nu k^2 F = \Psi \\ \Psi \text{ is a function of } k \text{ and of } t, \text{ infinitely small as } k^6 \\ \quad \text{and such that } \int_0^\infty \Psi \, dk = 0 \end{cases}$$

The following symbols and abbreviations are used:

X^* is the conjugate complex of X.

$\overline{u}$ is the mean value of u. Beginning with chapter II the mean values refer to a point, considered in the first chapter as conditional averages.

The bold-faced letters in the first paragraph, such as K, represent operators.

For simplifying the writing, notations such as $f(u; x)$ are used in place of what should be written unabbreviatedly $f(u_1,u_2,u_3; x_1,x_2,x_3)$; likewise, $\int f(u; x)$ is the abbreviation for $\iiint f(u_1,u_2,u_3; x_1,x_2,x_3)du_1\, du_2\, du_3$.

The formulas are numbered in every section; for example, formula (15-4) is the fourth formula in section 15.

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APPENDIX

Some experimental results.

In the foregoing analytical study of the various recent theories on turbulence, reference was made frequently to experimental results, mostly of English origin, but numerical values were given only rarely. The data contained in this appendix are intended to remedy this to some extent, by way of typical examples of several correlation curves and spectrum curves obtained by direct observations in the wind tunnel.

These curves have been determined by A. Favre and his collaborators at the Laboratory of Mechanics of the Atmosphere of the Institute of Fluid Mechanics, at Marseille, for the official French Aeronautical Research Establishment (O. N. E. R. A.). They are copied from reports by A. Favre,[13] except the spectral curve F(k) (fig. 20) which was calculated from other data.

The turbulent velocity was measured with considerably modified hot-wire instruments of the Dätwyler type which had undergone important modifications at the Marseille laboratory.

The time-correlation curve was obtained by means of a recording device with Tolana magnetic tape, remodeled in the laboratory so that it could be used for aerodynamic purposes. The electric current from the hot wire is recorded at a point on the magnetic tape driven at uniform forward speed V. Then the record is read at two points separated by a distance D, before which the tape unrolls. The currents obtained correspond to two turbulence records separated by a time displacement $\frac{D}{V}$ and then conjugated, as customary in space-correlation measurement, for the two currents coming simultaneously from two distinct hot wires. The method also makes it possible to measure the correlations with time and space displacement. So, from the experimental point of view, the problem of time correlations is solved. In this respect experiment is ahead of theory which up to now provides no clear prediction about the time-correlation curves. It is to be noted that the correlations at a point with time displacement are more precise to measure than the space correlations, because only one hot wire is used, and the time displacement can be diminished as desired, with the employed instrumentation. By way of contrast, for the space-correlations, it is difficult to get two perfectly identical hot wires, and it is impossible to bring them closer together beyond a certain limit.

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[13]A. Favre: Statistical time-correlation measurements; VII. International Congress of Applied Mechanics, London, 1948.

New time-space correlation measurements downstream from a turbulence grid, O. N. E. R. A. Report No. 8/522-A, Dec. 31, 1949.

All the curves reproduced here correspond to an overall speed of 12.2 m/sec. For the time-correlation curve (fig. 18) the intensity of turbulence was 1.92×10^{-2}.

The grid producing the turbulence in figure 16 was of 3.25 inch (8.3cm) mesh width; this value was chosen in order to facilitate the comparison with H. L. Dryden's measurements. In the other graphs a grid of 1 inch was involved. In all the cases the spacing of the hot wires from the grid was 40 meshes.

Figure 15 represents a space-correlation curve. The distance r of the hot wires, which were placed on a single horizontal perpendicular to the tunnel axis, serves as abscissa, Kármáns's dimensionless function $g(r)$ as ordinate.

Figure 16 represents the same function $g(r)$ for a different grid. The crosses correspond to a direct measurement; the round points correspond to the same measurement, but after recording on the time-correlation instrument, and rectification with zero time displacement.

This produces a certain check on the accuracy of the time-correlation measurements. The curve was extended up to near 200 mm. It remained reasonably constant and below the axis of the r.

Figure 17 shows the representative curve of the longitudinal correlation function $f(r)$, computed from $g(r)$ by Kármán's formula, and the curve representative of the function $\frac{1}{3}(f + 2g)$, proportional to the function $R(r)$, and obtained by calculation from the curve f and the curve g of figure 16. The values of R and f are very small and negative starting from $r = 120$ mm and $r = 200$ mm, respectively. (Compare formula (18-12).)

Figure 18 represents a time-correlation curve. It is seen to be strongly negative for settings above 5 milliseconds.

Figure 19 represents the spectral function $\underline{F}(n)$ of turbulence obtained by transformation of the time-correlation curves. It is therefore the Fourier transform in cosines of the time-correlation curve.[14]

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[14]More precisely, according to Taylor's formulas, $\underline{F}(n)$ is defined so that, if τ is the time displacement, $\int_0^\infty \underline{F}(n) \cos 2\pi n\tau \, dn$ is the time-correlation function, and, in particular, so that $\int_0^\infty \underline{F}(n)dn = 1$. In this case, infinity corresponds practically to 1,000 periods per second.

It was limited to the low frequencies of around 16 periods per second. The theory, as far as it can be applied to the spectral energy distribution curves in time at a point, suggests that the spectral curve passes through the origin. This corresponds to the fact that the areas of the positive and negative parts of the time correlation curve are equal. It is verified approximately on figure 18. This verification is not altogether rigorous, but it does seem that for values of more than 30 milliseconds in time displacement the curve becomes positive again, although remaining very close to the time axis.

Figure 20 gives, computed from the curves of figure 17, the integral

$$\frac{1}{\pi}\int_0^\infty kr \sin kr \frac{f(r) + 2g(r)}{3}\, dr$$

equal to the quotient of the spectral function $F(k)$ by the constant $3u_0^2$. Therefore, with suitable units as ordinates, it is the representative curve of $F(k)$; k is the inverse of a length, while in figure 19, n is the inverse of a time interval. A priori, Taylor's spectral function $\underline{F}(n)$, which represents the spectral energy distribution in time, and Heisenberg's spectral function $F(k)$ which represents the spectral energy distribution in space, are different.

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Figure 1.- Recording of the turbulent fluctuation velocity in a wind tunnel of 20 cm × 30 cm. Main-flow velocity: 20 m/sec. Intensity of turbulence: $5\text{-}10^{-3}$. The recording duration is 0.03 sec. (Photograph Heubès.)

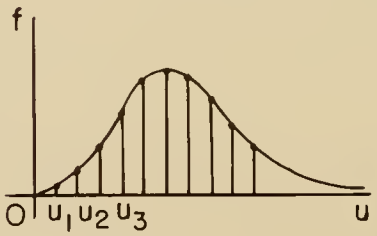
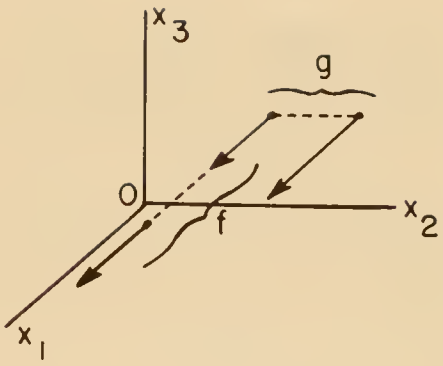


Figure 2



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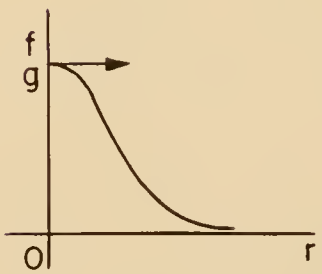

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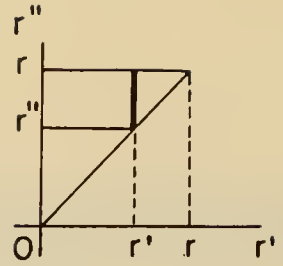

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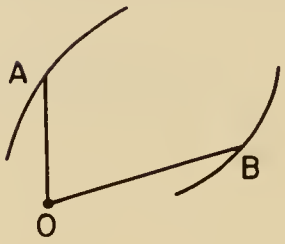

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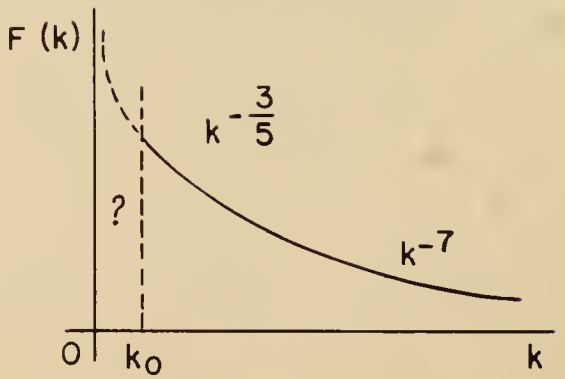

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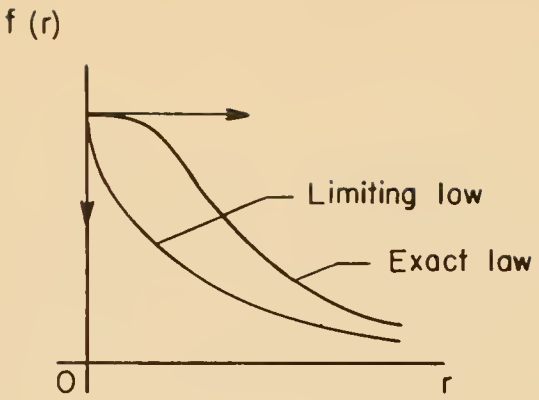

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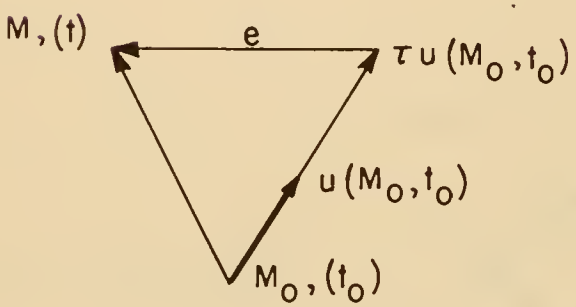

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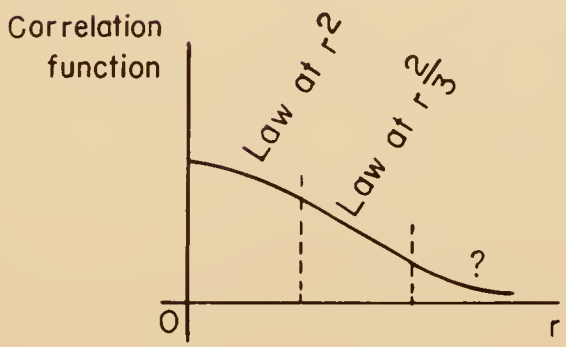

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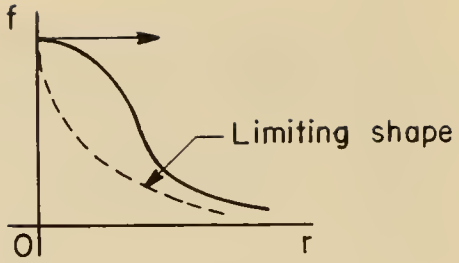

Figure 11

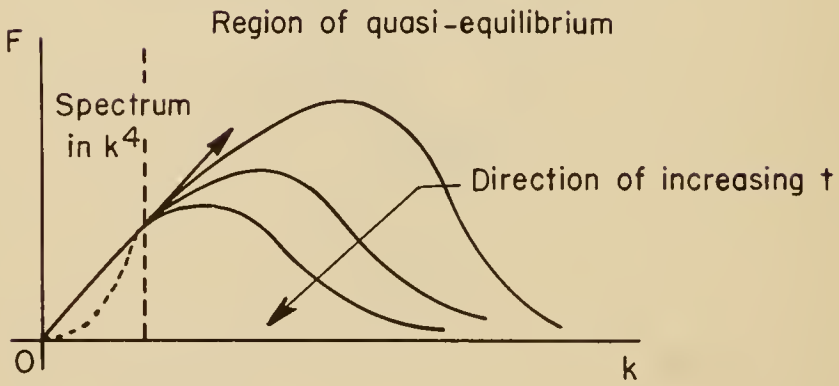

Figure 12

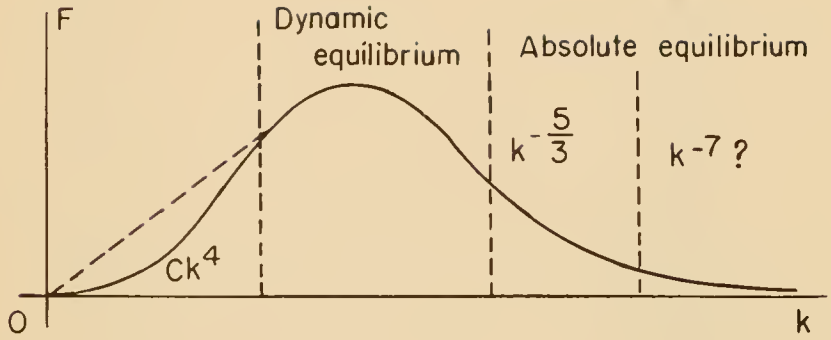

Figure 13

Figure 14

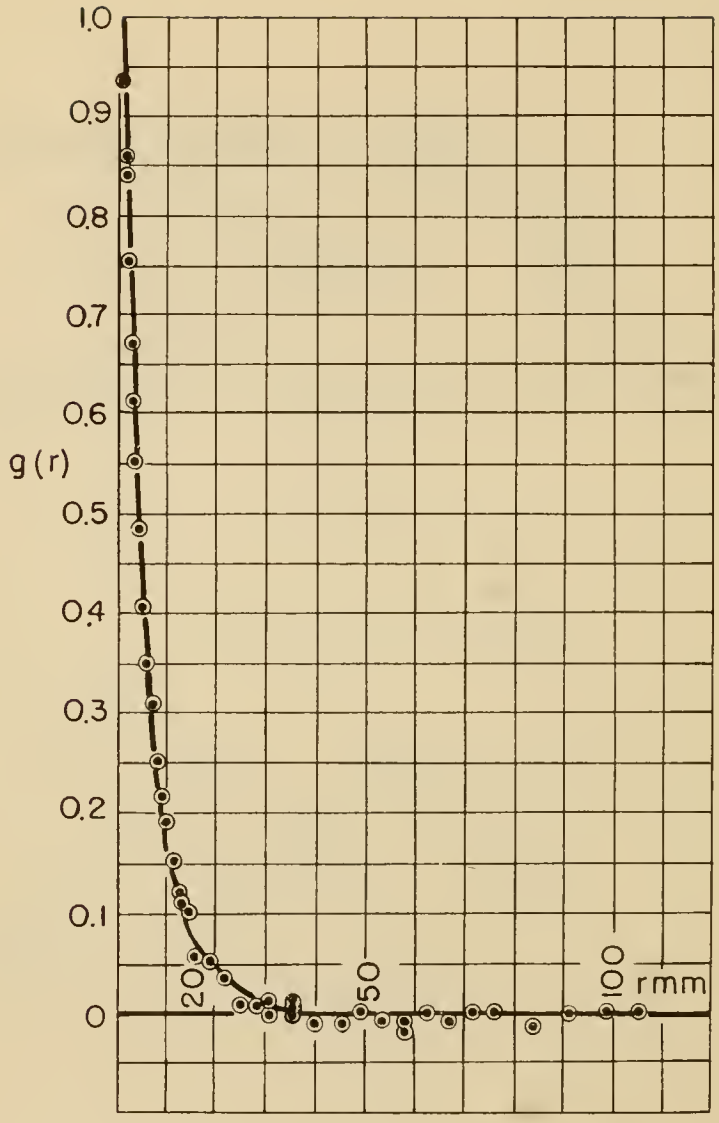

Figure 15.- Transverse correlation function g(r) in space.

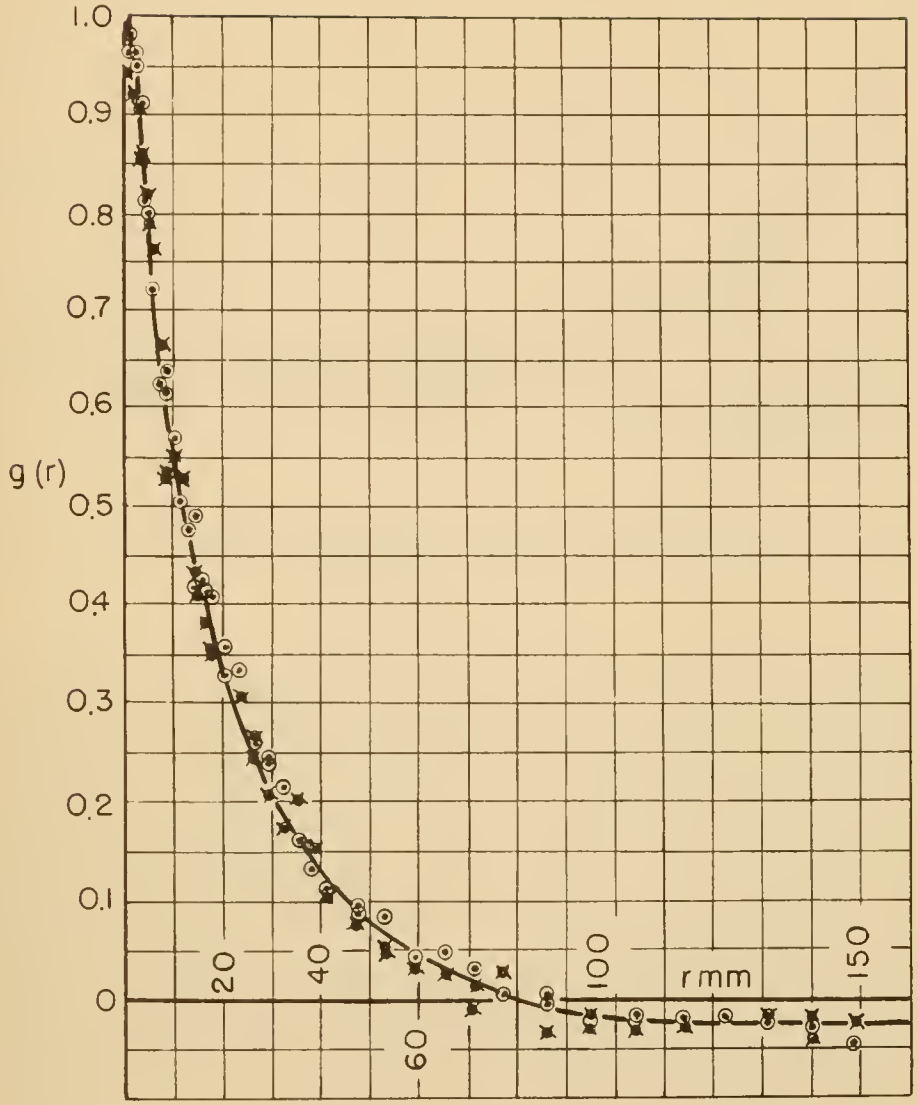

Figure 16.- Transverse correlation function g(r) in space.

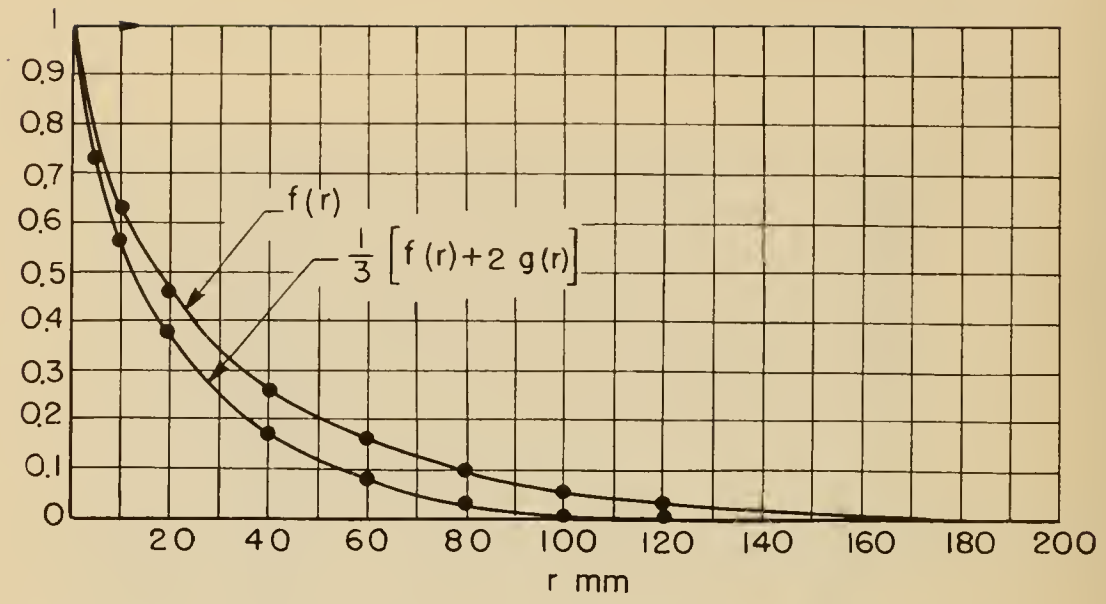

Figure 17.- Longitudinal correlation function f(r). Total correlation function $\frac{f(r) + 2g(r)}{3}$.

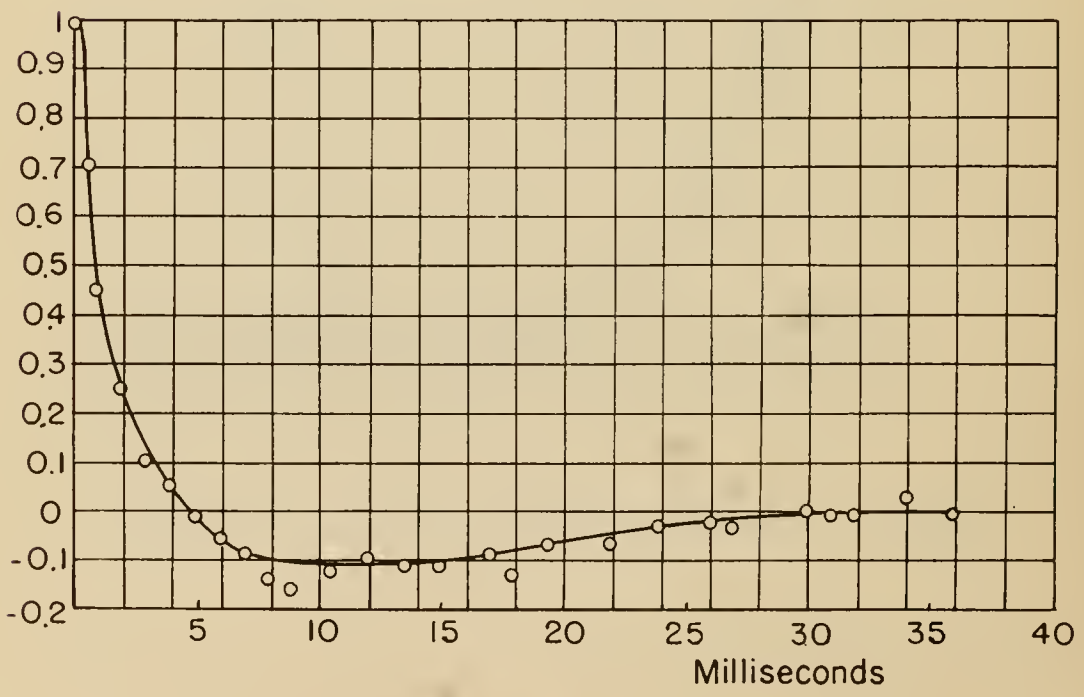

Figure 18.- Time correlation function.

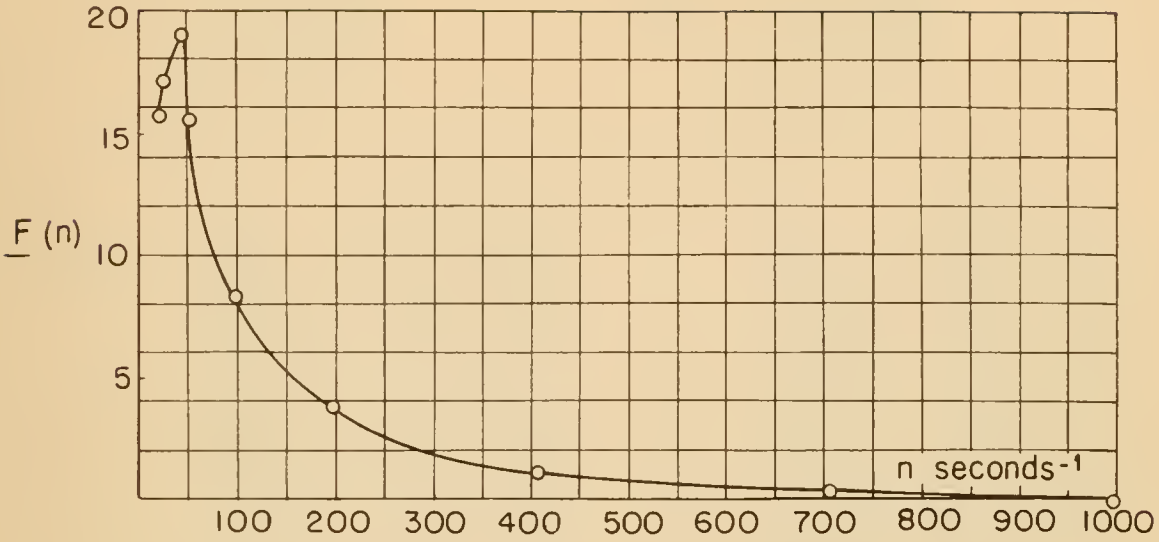

Figure 19.- Taylor's spectral function $\underline{F}(n)$.

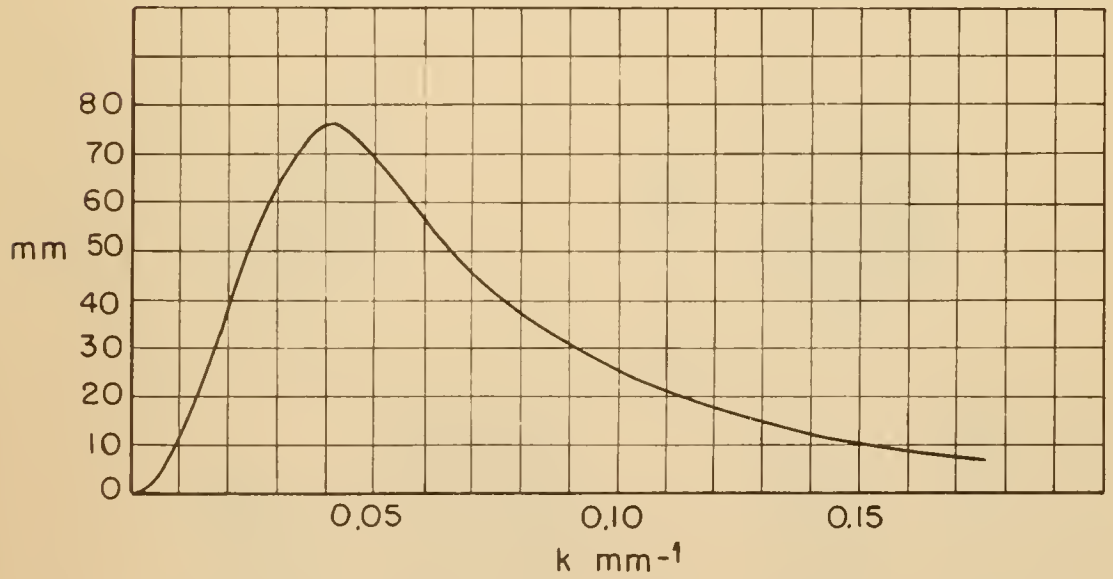

Figure 20.- Spectral function F(k). $\left(\text{Value of the integral } \frac{1}{\pi}\int_0^\infty kr \sin kr \frac{f + 2g}{3} dr\right)$.

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National Advisory Committee for Aeronautics.
THE THEORIES OF TURBULENCE. (Les Théories de la Turbulence). L. Agostini and J. Bass.
October 1955. 163p. diagrs. (NACA TM 1377. Trans. from Ministère de l'Air, Publications Scientifiques et Techniques 237, 1950)

The report includes a discussion of the kinematics of statistical mediums, particularly those which are isotropic. A mathematical study is made of the applications of Navier's equations to turbulent motion. Physical theories involving similarity are dealt with. Review is made of much of the work in turbulence. The theoretical discussions are illustrated by some correlation and spectrum curves based on measurements taken in the wind tunnel at the laboratory of the mechanics of the atmosphere at Marseille.

Copies obtainable from NACA, Washington

1. Flow, Turbulent (1.1.3.2)
I. Agostini, Leon
II. Bass, J.
III. NACA TM 1377
IV. Ministere de l'Air. Publications Scientifiques et Techniques 237, 1950

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